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ON THE IMPURITIES CONCENTRATIONS DYNAMICS IN MULTICOMPONENT SEMIDISPERSE SEMICOLLOID-HIGH-MOLECULAR SYSTEM: A CASE STUDY OF TWO-LAYERED ANISOTROPIC PEAT BLOCK

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Abstract. In the present paper, we propose an analytical approach for solving the 3D unsteady-state boundary-value problem for the second-order parabolic equation with the second and third type boundary conditions in two-layer rectangular parallelepiped domain. This problem is stimulated by the determining the dynamics of the concentration of metal substances in a two-layer anisotropic peat block. The work examines in detail the well-known variables separation method for constructing an analytical solution for a mathematical model of the studied problem. It is shown that the main difficulty is only the solution of the interrelated auxiliary problems, obtained from the original mathematical model under the certain conditions.

Keywords: unsteady-state diffusion equation, initial-boundary value problem, Robin boundary condition, analytical method.

AMS Subject Classification: 35K20, 93A30, 58J35, 76S05

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1 Introduction

In the present paper, which is an extended and appreciably augmented version of the work Guseynov et al. (2019), we propose an analytical approach for solving the 3D unsteady-state boundary-value problem for the second-order parabolic equation with the third type boundary conditions in two-layer rectangular parallelepipedic domain. Such type problems arise in particular at study of metal concentration dynamics in the peat blocks (for instance, see Teirumnieka et al. (2015); Kangro et al. (2014); Teirumnieka et al. (2011); Orru & Orru (2006); Brown et al. (2000) and respective references given in these).

Mathematical statement of the considered problem is taken from the article Teirumnieka et al. (2015), where the problem was solved by combination of the two approaches: firstly, the averaging method in the vertical direction (i.e. in height) and two horizontal directions (i.e. in width and in length), and, then, the obtained 2D problems have been solved by the standard/classical analytical methods. As opposed to the combinational approach suggested in Teirumnieka et al. (2015), in the present paper, we do not use approximation methods at all (basically, result of application of the averaging method always is approximate).

2 Mathematical formulation of problem

Denote by Ω_{ix} i-th $(i = 1,\, 2)$ layer of two-layer peat block, which has shape of rectangular parallelepiped (see Fig. 1):

$$\begin{gathered}\Omega_{ix} \overset{def}{\equiv} \left\{ x = (x_1,\, x_2,\, x_3) \in \mathbb{R}^3 \right| \ \ x_j \in [0,\, L_j]\,,\ j = 1,\, 2; \\ (i-1)\, H_1 \leq x_3 \leq (i-1)\,(L_3 - H_1) + H_1 \}\,,\ i = 1,\, 2.\end{gathered}$$

Now we formulate a mathematical model describing the dynamics of metal concentration in a two-layer peat block: it is required to find functions $c_i\,(x,\, t) : \Omega_{ix} \times [0,\, t_{END}] \to \mathbb{R}^1,\ (i = 1,\, 2)\,,$ which satisfy

- diffusion equations with sources

$$\frac{\partial c_i(x,t)}{\partial t} = \sum_{j=1}^{3} D_{ij} \frac{\partial^2 c_i(x,t)}{\partial x_j^2} + f_i\,(x,\, t)\,,\ (x,\, t) \in \mathrm{int}\Omega_{ix} \times (0,\, t_{END}]\,,\ (i = 1,\, 2)\,; \tag{1}$$

- initial conditions

$$c_i\,(x,\, t)|_{t=0^+} = c_{i0}\,(x)\,,\ x \in \Omega_{ix},\ (i = 1,\, 2)\,; \tag{2}$$

- the following boundary conditions given:
  - at the trailing wall in the form of von Neumann condition

$$\begin{gathered}\left.\frac{\partial c_i(x,t)}{\partial x_1}\right|_{x_1=0^+} = c_{i1}\,(x_2,\, x_3,\, t)\,, \\ (x/\{x_1\},\, t) \in \Omega_{ix/\{x_1\}} \times [0,\, t_{END}]\,,\ (i = 1,\, 2)\,;\end{gathered} \tag{3}$$

  - at the front wall in the form of Robin condition

$$\begin{gathered}\left.\left[D_{i1} \frac{\partial c_i(x,t)}{\partial x_1} + \lambda_{i1} c_i\,(x,\, t)\right]\right|_{x_1=L_1^-} = a_{i1}\,(x_2,\, x_3,\, t)\,, \\ (x/\{x_1\},\, t) \in \Omega_{ix/\{x_1\}} \times [0,\, t_{END}]\,,\ (i = 1,\, 2)\,;\end{gathered} \tag{4}$$

  - at the left-side wall in the form of von Neumann condition

$$\begin{gathered}\left.\frac{\partial c_i(x,t)}{\partial x_2}\right|_{x_2=0^+} = c_{i2}\,(x_1,\, x_3,\, t)\,, \\ (x/\{x_2\},\, t) \in \Omega_{ix/\{x_2\}} \times [0,\, t_{END}]\,,\ (i = 1,\, 2)\,;\end{gathered} \tag{5}$$

  - at the right-side wall in the form of Robin condition

$$\begin{gathered}\left.\left[D_{i2} \frac{\partial c_i(x,t)}{\partial x_2} + \lambda_{i2} c_i\,(x,\, t)\right]\right|_{x_2=L_2^-} = a_{i2}\,(x_1,\, x_3,\, t)\,, \\ (x/\{x_2\},\, t) \in \Omega_{ix/\{x_2\}} \times [0,\, t_{END}]\,,\ (i = 1,\, 2)\,;\end{gathered} \tag{6}$$

  - at the lower (by $i = 1$) and the upper (by $i = 2$) bases in the form of Robin condition

$$\begin{gathered}\left.\left[D_{i3} \frac{\partial c_i(x,t)}{\partial x_3} + (2i-3)\,\lambda_{i3} c_i\,(x,\, t)\right]\right|_{x_3=(i-1)^+ L_3^-} = a_{i3}\,(x_1,\, x_2,\, t)\,, \\ (x/\{x_3\},\, t) \in \Omega_{ix/\{x_3\}} \times [0,\, t_{END}]\,,\ (i = 1,\, 2)\,;\end{gathered} \tag{7}$$

- matching conditions given at the bedding interface

$$\begin{gathered}((i-1)\,(D_{13}-1)+1)\ \left.\frac{\partial^{i-1} c_1(x,t)}{\partial x_3}\right|_{x_3=H_1^-} = ((i-1)\,(D_{23}-1)+1)\ \left.\frac{\partial^{i-1} c_2(x,t)}{\partial x_3}\right|_{x_3=H_1^+}, \\ (x/\{x_3\},\, t) \in \Omega_{ix/\{x_3\}} \times [0,\, t_{END}]\,,\ (i = 1,\, 2)\,;\end{gathered} \tag{8}$$

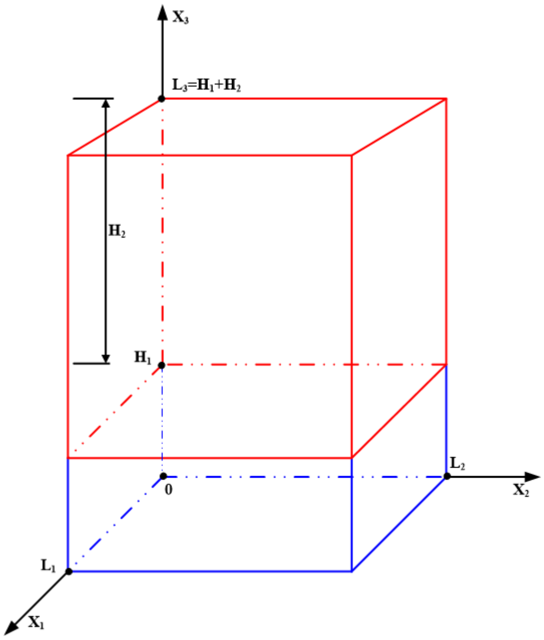

Figure 1: Schematic representation of two-layer peat block in the form of rectangular parallelepipedic domain.

- all 12 consistency conditions linking the initial and boundary functions from the constraints (2)-(7): such as $\left.\frac{\partial c_{i0}(x)}{\partial x_1}\right|_{x_1=0^+} = c_{i1}\left(x_2,\, x_3,\, 0\right),\ \left.\frac{\partial c_{i0}(x)}{\partial x_1}\right|_{x_1=0^+} = c_{i1}\left(x/\{x_1\},\, 0\right),$ etc.

In the mathematical model (1)-(8) it is assumed that all numerical parameters $L_i > 0\ \left(i = \overline{1,\, 3}\right),$ $H_i > 0\ \ (i = 1,\, 2)\,,\ D_{ij} > 0\ \left(i = 1,\, 2;\ j = \overline{1,\, 3}\right),\ \lambda_{ij} > 0\ \left(i = 1,\, 2;\ j = \overline{1,\, 3}\right),\ t_{END} > 0$, and all functions except functions $c_1\left(x,\, t\right)$ and $c_2\left(x,\, t\right),$ which stand for the desired metal concentrations, respectively, in the first and second layers of the peat block, are a priori given.

In (1)-(8) some specific denotations are used, whose meaning is explained below: $\Omega_{ix/\{x_j\}} = \Omega_{ix}/\{x_j\}$ represents the corresponding face (when $x_j = 0$) of i-th $(i = 1,\, 2)$ layer of two-layer peat block; $\mathrm{int}\Omega_{i,\, x}$ means the interior of the set $\Omega_{i,\, x}$; denotations $\left.g\left(y\right)\right|_{y=A^-}$ and $\left.g\left(y\right)\right|_{y=A^+}$ should be understood respectively as the left and right limits of the function $g\left(y\right)$ at point $y = A$, i.e. $\left.g\left(y\right)\right|_{y=A^{\mp}} = \lim\limits_{y\to A^{\mp}} g\left(y\right).$

Remark 1. *If in the mathematical model* (1)-(8) *we assume that:*

(a) $c_{i1}\left(\bullet\right) = c_{i2}\left(\bullet\right) \equiv 0,\ \ (i = 1,\, 2)\,;$

(b) boundary functions $a_{ij}\left(\bullet\right),\ \left(i = 1,\, 2;\ j = \overline{1,\, 3}\right)$ *do not depend on time* t*,*

then the model (1)-(8) *will completely coincide with the mathematical model (1.1) from Teirumnieka (2015), in which the physical interpretations of all the initial data - numerical parameters and functions are exhaustively described. Therefore, in this paper we will not describe the physical meaning of the initial numerical parameters and functions of the model* (1)-(8)*: they have the same meaning as in Teirumnieka et al. (2015).*

Remark 2. *Let us pay attention to the initial conditions* (2)*, which mean that in each layer the initial concentration of impurities is distributed by its own distinctive law. Such a case is an atypical case, since when modeling and studying the majority of dynamic transport processes in heterogeneous media with lumped factors, it is usually assumed that the distribution of the initial concentration (or initial temperature, etc.) is subordinate to the same law for all layers of the considered layered medium (both anisotropic medium and isotropic medium). The above* *indicated non-typical initial condition is the first essential feature of the problem considered in Teirumnieka et al. (2015) and in this paper. The second feature (less significant) of this problem is that both layers of fine-pore medium are anisotropic layers, each layer having its own distinctive anisotropy. Finally, we note that if the problem under consideration did not have these two features, then it would be a trivial problem, finding an analytical solution of which is studied in the framework of the usual course "Equations of Mathematical Physics" for students (Tikhonov & Samarsky, 1990).*

3 On two approaches to solve the stated problem using an analytical method

The mathematical model (1)-(8) can be solved by two different approaches. The first approach is a more universal approach for solving wide classes of initial-boundary value problems in layered regions with layers, whose physical, chemical, etc. characteristics are different. The main idea of the first approach is to perform the following procedures:

Procedure 1. By artificially introducing a non-existent/missing boundary condition of the first kind (or of the second kind) at the upper boundary of the first layer Ω_{1x} (i.e. at point $x_3 = H_1^-$), for example, like

$$c_1(x,\,t)\big|_{x_3=H_1^-} = u(x_1,\,x_2,\,t)\,,\ (x_1,\,x_2,\,t)\in\Omega_{1x/\{x_3\}}\times[0,\,t_{END}]\,,$$

we obtain complete initial-boundary value problem for the desired function $c_1(x,\,t)$, the solution of which by the Green's function method is easy to express in an analytical form containing yet an unknown function $u(x,\,t):\Omega_{1x/\{x_3\}}\times[0,\,t_{END}]\to\mathbb{R}$, i.e. we have $c_1(x,\,t) = \Xi_1(\ldots,\,(Au(x_1,\,x_2,\,t)))$, where A is a corresponding Fredholm integral operator.

Procedure 2. Similarly, by artificially introducing a non-existent/missing boundary condition of the first kind (or, respectively, of the second kind) at the lower boundary of the second layer $\Omega_{2,x}$ (i.e. at point $x_3 = H_1^+$), for example, like

$$c_2(x,\,t)\big|_{x_3=H_1^+} = \vartheta(x_1,\,x_2,\,t)\,,\ (x_1,\,x_2,\,t)\in\Omega_{2x/\{x_3\}}\times[0,\,t_{END}]\,,$$

we will have a complete initial-boundary value problem for desired function $c_2(x,\,t)$, the solution of which by the Green's function method is also easy to express in an analytical form containing yet an unknown function $\vartheta(x,\,t):\Omega_{2x/\{x_3\}}\times[0,\,t_{END}]\to\mathbb{R}^1$, i.e. we have $c_2(x,\,t) = \Xi_2(\ldots,\,(B\vartheta(x_1,\,x_2,\,t)))$, where B is a corresponding Fredholm integral operator.

Procedure 3. Since matching conditions (1) take place, we find that in the previous two procedures, artificially introduced boundary functions $u(x,\,t)$ and $\vartheta(x,\,t)$, are connected by relation

$$u(x,\,t) = \vartheta(x,\,t)\,, \tag{9}$$

if artificially introduced boundary conditions are conditions of the first kind; or by relation

$$D_{13}u(x,\,t) = D_{23}\vartheta(x,\,t)\,, \tag{10}$$

if artificially introduced boundary conditions are conditions of the second kind.

Procedure 4. Since taking into account (9) or (10) we have $c_1(x,\,t) = \Xi_1(\ldots,\,(Au(x_1,\,x_2,\,t)))$, $c_2(x,\,t) = \Xi_2(\ldots,\,(Bu(x_1,\,x_2,\,t)))$, the use of the matching condition from (8), which was not used in establishing the relationship (9) or (10), leads to the Fredholm integral equation of the first kind for finding the artificially introduced boundary function $u(x,\,t)$. Applying the Tikhonov regularization method (Tikhonov& Arsenin, 1977; Andreyev & Guseynov, 2013) to the obtained Fredholm integral equation of the first kind, we find its regularized solution $u(x,\,t) = u_{\text{Reg.}}(x_1,\,x_2,\,t)$, where $x_1\in[0,\,L_1]$, $x_2\in[0,\,L_2]$, $t\in[0,\,t_{END}]$. Finally, taking into account the found solution $u(x,\,t) = u_{\text{Reg.}}(x_1,\,x_2,\,t)$ in analytical expressions for $c_1(x,\,t)$ and

$c_2(x, t)$ completely determines the desired functions: $c_1(x, t) = \Xi_1(\ldots, (Au_{\text{Reg.}}(x_1, x_2, t)))$, $c_2(x, t) = \Xi_2(\ldots, (Bu_{\text{Reg.}}(x_1, x_2, t)))$.

Remark 3. *As is was already mentioned at the beginning of this section, the above described approach, consisting of procedures 1-4, is a more universal approach, and this universality lies in its sufficiently wide applicability to the most diverse problems of mathematical physics, in which the considered region is a heterogeneous medium. At the same time, as we have seen, in the course of applying this approach, one has to solve an inverse ill-posed problem: in our case, this problem is the Fredholm integral equation of the first kind. It seems to us that this circumstance is the reason for the relatively little knowledge and rare applications of this elegant approach to problems of this kind.*

The essence of the second approach consists of applying method of separation of variables and constructing the Green's function (for instance, see (Tikhonov & Samarsky, 1990) that is one of the best mathematical textbooks ever written). This method of solving initial problems (a Cauchy problem, when the region in which the process is studied is an unbounded region), boundary-value problems (in the case when the steady-state process is studied, or the process is studied at a time sufficiently far from the initial moment of the process) and initial-boundary value problems is a more "traditional" approach in the sense that this technique is, firstly, thoroughly studied in almost all courses of equations of mathematical physics and/or partial differential equations, and, secondly, is widely used in the study of various kinds of mathematical models described in the language of initial, boundary and initial-boundary problems for partial differential equations, in particular, for hyperbolic, parabolic and elliptic types of differential equations. In this paper, the second approach is chosen as the analytical method for solving the mathematical model (1)-(8) – the method of separation of variables and construction of the corresponding Green's function.

4 Application of the method of separation of variables, and construction of the corresponding Green's functions

So, let us consider the initial-boundary value problem (1)-(8), and try to find its solution by applying the method of separation of variables. To do this, we first formulate, as is customary in the method of separation of variables (Tikhonov & Samarsky, 1990), two auxiliary boundary-value problems – the problem AP1 and the problem AP2, in each of which the equation is homogeneous.

4.1 Formulation of two auxiliary boundary-value problems

<u>Auxiliary problem AP1</u>. It is required to find the function $0 \not\equiv c_1(x, t) : \Omega_{1x} \times [0, t_{END}] \to \mathbb{R}^1$ that satisfies:

- homogeneous equation

$$\frac{\partial c_1(x, t)}{\partial t} = \sum_{j=1}^{3} D_{1j} \frac{\partial^2 c_1(x, t)}{\partial x_j^2}, \quad (x, t) \in \mathrm{int}\Omega_{1x} \times (0, t_{END}], \tag{11}$$

- heterogeneous initial condition

$$c_1(x, t)|_{t=0^+} = c_{10}(x), \; x \in \Omega_{1x}, \tag{12}$$

- homogeneous boundary conditions

$$\left.\frac{\partial c_1(x, t)}{\partial x_1}\right|_{x_1=0^+} = 0, \; (x/\{x_1\}, t) \in \Omega_{1x/\{x_1\}} \times [0, t_{END}], \tag{13}$$

$$D_{11}\left.\frac{\partial c_1(x,\,t)}{\partial x_1}\right|_{x_1=L_1^-}+\left.\lambda_{11}c_1(x,\,t)\right|_{x_1=L_1^-}=0,\;(x/\{x_1\},\,t)\in\Omega_{1x/\{x_1\}}\times[0,\,t_{END}]\,,\quad(14)$$

$$\left.\frac{\partial c_1(x,\,t)}{\partial x_2}\right|_{x_2=0^+}=0,\;(x/\{x_2\},\,t)\in\Omega_{1x/\{x_2\}}\times[0,\,t_{END}]\,,\quad(15)$$

$$D_{12}\left.\frac{\partial c_1(x,\,t)}{\partial x_2}\right|_{x_2=L_2^-}+\left.\lambda_{12}c_1(x,\,t)\right|_{x_2=L_2^-}=0,\;(x/\{x_2\},\,t)\in\Omega_{1x/\{x_2\}}\times[0,\,t_{END}]\,,\quad(16)$$

$$D_{13}\left.\frac{\partial c_1(x,\,t)}{\partial x_3}\right|_{x_3=0^+}-\left.\lambda_{13}c_1(x,\,t)\right|_{x_3=0^+}=0,\;(x/\{x_3\},\,t)\in\Omega_{1x/\{x_3\}}\times[0,\,t_{END}]\,,\quad(17)$$

- as well as two conditions

$$\left.c_1(x,\,t)\right|_{x_3=H_1^-}=\left.c_2(x,\,t)\right|_{x_3=H_1^+},\;(x_1,\,x_2,\,t)\in[0,\,L_1]\times[0,\,L_2]\times[0,\,t_{END}]\,,\quad(18)$$

$$D_{13}\left.\frac{\partial c_1(x,\,t)}{\partial x_3}\right|_{x_3=H_1^-}=D_{23}\left.\frac{\partial c_2(x,\,t)}{\partial x_3}\right|_{x_3=H_1^+},\;(x_1,\,x_2,\,t)\in[0,\,L_1]\times[0,\,L_2]\times[0,\,t_{END}]\,,\quad(19)$$

where the function $c_2(x,\,t)$ is defined in the domain $\Omega_{2x}\times[0,\,t_{END}]$ and is a nontrivial solution of the problem AP2 stated below.

Auxiliary problem AP2. It is required to find the function $0\not\equiv c_2(x,\,t):\Omega_{2x}\times[0,\,t_{END}]\to\mathbb{R}^1$ that satisfies:

- homogeneous equation

$$\frac{\partial c_2(x,\,t)}{\partial t}=\sum_{j=1}^{3}D_{2j}\frac{\partial^2 c_2(x,\,t)}{\partial x_j^2},\;(x,\,t)\in\mathrm{int}\Omega_{2x}\times(0,\,t_{END}]\,,\quad(20)$$

- heterogeneous initial condition

$$\left.c_2(x,\,t)\right|_{t=0^+}=c_{20}(x)\,,\;x\in\Omega_{2x},\quad(21)$$

- homogeneous boundary conditions

$$\left.\frac{\partial c_2(x,\,t)}{\partial x_1}\right|_{x_1=0^+}=0,\;(x/\{x_1\},\,t)\in\Omega_{2x/\{x_1\}}\times[0,\,t_{END}]\,,\quad(22)$$

$$D_{21}\left.\frac{\partial c_2(x,\,t)}{\partial x_1}\right|_{x_1=L_1^-}+\left.\lambda_{21}c_2(x,\,t)\right|_{x_1=L_1^-}=0,\;(x/\{x_1\},\,t)\in\Omega_{2x/\{x_1\}}\times[0,\,t_{END}]\,,\quad(23)$$

$$\left.\frac{\partial c_2(x,\,t)}{\partial x_2}\right|_{x_2=0^+}=0,\,(x/\{x_2\},\,t)\in\Omega_{2x/\{x_2\}}\times[0,\,t_{END}]\,,\quad(24)$$

$$D_{22}\left.\frac{\partial c_2(x,\,t)}{\partial x_2}\right|_{x_2=L_2^-}+\left.\lambda_{22}c_1(x,\,t)\right|_{x_2=L_2^-}=0,\;(x/\{x_2\},\,t)\in\Omega_{2x/\{x_2\}}\times[0,\,t_{END}]\,,\quad(25)$$

$$D_{23}\left.\frac{\partial c_2(x,\,t)}{\partial x_3}\right|_{x_3=L_3^-}+\left.\lambda_{23}c_2(x,\,t)\right|_{x_3=L_3^-}=0,\;(x/\{x_3\},\,t)\in\Omega_{2x/\{x_3\}}\times[0,\,t_{END}]\,.\quad(26)$$

Remark 4. *As it was noted in Remark 2, in the problem studied in this work (and also in Teirumnieka et al. (2015)), each of the two layers has its own distinctive law of distribution of impurities concentration at the initial moment of time* $t=0$*. That is why, in the auxiliary problem AP2, the initial condition* (21) *is present: if the distribution of the initial impurity concentration for both layers obeys the same law, i.e.* $\left.c(x,\,t)\right|_{t=0^+}=c_0(x)\,,\;\forall x\in\Omega_{1x}\cup\Omega_{2x}$*, then in the auxiliary problem AP2 the initial condition* (21) *should not be present.*

4.2 Partial investigation of the first auxiliary boundary-value problem

First of all, we note that the use of the phrase “partial research” in the titles of the current and next subsections is related to the interconnectedness of the auxiliary problems AP1 and AP2: as it will be seen in subsections 4.2 and 4.3 of this section, a full research of AP1 is impossible without research of AP2, and vice versa.

So, first consider the AP1 problem, a non-trivial solution of which will be sought in the form

$$c_1(x,\, t) = T_1(t) \sum_{j=1}^{3} X_{1j}(x_j), \tag{27}$$

where the essence of the requirements $X_{1j}(x_j) \not\equiv 0,\ j = \overline{1,\, 3}$ and $T_1(t) \not\equiv 0$ is obvious.

Taking into account representation (27) in the equation (11), we get

$$\frac{T'_1(t)}{T_1(t)} = \sum_{j=1}^{3} D_{1j} \frac{X''_{1j}(x_j)}{X_{1j}(x_j)}. \tag{28}$$

Since the left side of equality (28) depends only on the time variable t and the right side depends only on spatial variables $x = (x_1,\, x_2,\, x_3)$, equality (28) is possible only if both sides of it are equal to the same constant, which we will denote by $-\mu_1$, without making any assumptions regarding the sign of the constant μ_1.

So, instead of (28) we can write the following two equations:

$$T'_1(t) + \mu_1 T_1(t) = 0, \tag{29}$$

$$\sum_{j=1}^{3} D_{1j} \frac{X''_{1j}(x_j)}{X_{1j}(x_j)} + \mu_1 = 0. \tag{30}$$

First we deal with equation (30), and then we return to equation (29). Alternately differentiating equation (30) with respect to variables x_1, x_2 and x_3, we obtain

$$\frac{d}{dx_j}\left(D_{1j} \frac{X''_{1j}(x_j)}{X_{1j}(x_j)} \right) = 0,\ \forall j = \overline{1,\, 3},$$

from which it follows that $D_{1j}\frac{X''_{1j}(x_j)}{X_{1j}(x_j)},\ \forall j = \overline{1,\, 3}$ are constants: $D_{1j}\frac{X''_{1j}(x_j)}{X_{1j}(x_j)} = \mu_{1j},\ \forall j = \overline{1,\, 3}$, where $\mu_1 = \sum\limits_{j=1}^{3} \mu_{1j}$, i.e. new constants μ_{11}, μ_{12}, μ_{13} are constituent constants of the initial constant μ_1, appearing in equations (29) and (30).

So, we obtained the following homogeneous equations of the same type:

$$D_{11} X''_{11}(x_1) + \mu_{11} X_{11}(x_1) = 0, \tag{31}$$

$$D_{12} X''_{12}(x_2) + \mu_{12} X_{12}(x_2) = 0, \tag{32}$$

$$D_{13} X''_{13}(x_3) + \mu_{13} X_{13}(x_3) = 0, \tag{33}$$

which are related only by the fact that $\mu_1 = \sum\limits_{j=1}^{3} \mu_{1j}$. As the constant’s μ_1 sign still is unknown to us, we also have no information about constants’ μ_{11}, μ_{12}, μ_{13} signs.

Further, the substitution of (27) to the boundary conditions (13)-(17) gives us the following boundary conditions:

- for the function $X_{11}(x_1)$ two boundary conditions:

$$\begin{cases} X'_{11}(0) = 0, \\ D_{11}X'_{11}(L_1) + \lambda_{11}X_{11}(L_1) = 0; \end{cases} \tag{34}$$

- for the function $X_{12}(x_2)$ again two boundary conditions:

$$\begin{cases} X'_{12}(0) = 0, \\ D_{12}X'_{12}(L_2) + \lambda_{12}X_{12}(L_2) = 0; \end{cases} \tag{35}$$

- for the function $X_{13}(x_3)$ one boundary condition:

$$D_{13}X'_{13}(0) - \lambda_{13}X_{13}(0) = 0. \tag{36}$$

Consequently, the combination of equation (31) and boundary conditions (34), the combination of equation (32) and boundary conditions (35), and finally the combination of equation (33) and boundary condition (36) generate the following three Sturm-Liouville problems (for instance, see (Al-Gwaiz, 2008)), the first two of which are complete problems (in the sense that they have a complete formulation: each of them has a second-order ordinary differential equation and two boundary conditions are given), and the third problem is an incomplete problem (one boundary condition is missing):

$$\begin{cases} D_{11}X''_{11}(x_1) + \mu_{11}X_{11}(x_1) = 0, \ x_1 \in (0, L_1), \\ X'_{11}(0) = 0, \\ D_{11}X'_{11}(L_1) + \lambda_{11}X_{11}(L_1) = 0; \end{cases} \tag{37}$$

$$\begin{cases} D_{12}X''_{12}(x_2) + \mu_{12}X_{12}(x_2) = 0, \ x_2 \in (0, L_2), \\ X'_{12}(0) = 0, \\ D_{12}X'_{12}(L_2) + \lambda_{12}X_{12}(L_2) = 0; \end{cases} \tag{38}$$

$$\begin{cases} D_{13}X''_{13}(x_3) + \mu_{13}X_{13}(x_3) = 0, \ x_3 \in (0, H_1), \\ D_{13}X'_{13}(0) - \lambda_{13}X_{13}(0) = 0. \end{cases} \tag{39}$$

We will have to solve all three Sturm-Liouville problems in turn (37)-(39): our goal is to find their non-trivial solutions $X_{1j}(x_j) \not\equiv 0, \ j = \overline{1, 3}$. We will show that in the case, when $\mu_{11} \le 0$, the problem (37) has only a trivial solution. Indeed,

- In the case, when $\mu_{11} = 0$, from (37) we obtain that $X_{11}(x_1) = Cx_1$, where C is a constant that must satisfy equality $D_{11}C + \lambda_{11}CL_1 = 0$. Since $D_{11} + \lambda_{11}L_1 > 0$, it is obvious, that $C = 0$ and, consequently, $X_{11}(x_1) \equiv 0, \ \forall x_1 \in [0, L_1]$.

- In the case, when from (37) we obtain that $X_{11}(x_1) = C_1 e^{-\sqrt{\frac{|\mu_{11}|}{D_{11}}}x_1} + C_2 e^{\sqrt{\frac{|\mu_{11}|}{D_{11}}}x_1}$, where C_1 and C_2 are constants that must satisfy equalities

$$\left.\begin{array}{l} C_1 + C_2 = 0, \\ C_1\left(\sqrt{|\mu_{11}|\, D_{11}} + \lambda_{11}\right) = e^{2\sqrt{\frac{|\mu_{11}|}{D_{11}}}L_1} C_2\left(\lambda_{11} - \sqrt{|\mu_{11}|\, D_{11}}\right). \end{array}\right\}$$

From these two equalities we obtain the identity

$$C_1\left(\frac{\lambda_{11} - \sqrt{|\mu_{11}|\, D_{11}}}{\lambda_{11} + \sqrt{|\mu_{11}|\, D_{11}}} - e^{2\sqrt{\frac{|\mu_{11}|}{D_{11}}}L_1}\right) = 0.$$

Since it always holds that $\frac{\lambda_{11}-\sqrt{|\mu_{11}|D_{11}}}{\lambda_{11}+\sqrt{|\mu_{11}|D_{11}}} < 1$ and $e^{2\sqrt{\frac{|\mu_{11}|}{D_{11}}}L_1} > 1$, and, therefore,

$$\frac{\lambda_{11}-\sqrt{|\mu_{11}|\,D_{11}}}{\lambda_{11}+\sqrt{|\mu_{11}|\,D_{11}}} - e^{2\sqrt{\frac{|\mu_{11}|}{D_{11}}}L_1} \neq 0,$$

from the last equality it follows that $C_1 = 0$. Therefore, $C_2 = -C_1 = 0$. Then we get that $X_{11}(x_1) = C_1 e^{-\sqrt{\frac{|\mu_{11}|}{D_{11}}}x_1} + C_2 e^{\sqrt{\frac{|\mu_{11}|}{D_{11}}}x_1} \equiv 0,\ x_1 \in [0,\, L_1]$.

So, in problem (37) only the case should be considered, and in this case the general solution of the problem (37) is the following function

$$X_{11}(x_1) = A_{11}\cos\left(\sqrt{\frac{\mu_{11}}{D_{11}}}x_1\right),\ x_1 \in [0,\, L_1], \tag{40}$$

where A_{11} is an arbitrary constant.

Function $X_{11}(x_1),\ x_1 \in [0,\, L_1]$, defined by formula (40), is called eigenfunction of the Sturm-Liouville problem (for instance, see (Al-Gwaiz, 2008) as well as (Tikhonov & Samarsky, 1990)), and it corresponds to the eigenvalue $\mu_{11} = D_{11}\left(\frac{\alpha_1}{L_1}\right)^2 > 0$, where α_1 is a positive root of the transcendental equation

$$\alpha_1 tg(\alpha_1) = \frac{\lambda_{11}L_1}{D_{11}}. \tag{41}$$

Since the transcendental equation (41) has an infinite number of solutions, we can say that the Sturm-Liouville problem (37) has an infinite number of eigenvalues

$$\mu_{11n} = D_{11}\left(\frac{\alpha_{1n}}{L_1}\right)^2 > 0, \tag{42}$$

to which the following eigenfunctions correspond

$$X_{11n}(x_1) = A_{11n}\cos\left(\sqrt{\frac{\mu_{11n}}{D_{11}}}x_1\right),\ x_1 \in [0,\, L_1], \tag{43}$$

and each of them is determined with precision to an arbitrary constant A_{11n}. In (43) number α_{1n} is n-th ($n \in \mathbb{N}$) positive root of the transcendental equation (41), and, hereinafter, speaking of the ordinal numbers of the positive roots of the transcendental equation, we will mean their ordering in non-decreasing order: $\alpha_{11} \le \alpha_{12} \le \alpha_{13} \le \ldots$.

Because of the fact that the Sturm-Liouville problem (38) differs from the problem (37) only by the coefficients D_{12} and λ_{12}, we can write, fully following the results of the study of the problem (37) obtained above, that the eigenfunctions of the Sturm-Liouville problem (38) are the functions

$$X_{12m}(x_2) = A_{12m}\cos\left(\sqrt{\frac{\mu_{12m}}{D_{12}}}x_2\right),\ x_2 \in [0,\, L_2], \tag{44}$$

where A_{12m} are some constants; $\mu_{12m} = D_{12}\left(\frac{\beta_{1m}}{L_2}\right)^2 > 0$ are eigenvalues, β_{1m} is m-th ($m \in \mathbb{N}$) positive root of the transcendental equation

$$\beta_1 tg(\beta_1) = \frac{\lambda_{12}L_2}{D_{12}}. \tag{45}$$

Now we investigate the incomplete Sturm-Liouville problem (39). It is easy to check that when $\mu_{13} > 0$ (in the case of $\mu_{13} \le 0$ problem (39) has only trivial solution) incomplete problem (39) has general solution

$$X_{13}(x_3) = A_{13}\left\{\sin\left(\sqrt{\frac{\mu_{13}}{D_{13}}}x_3\right) + \frac{\sqrt{\mu_{13}D_{13}}}{\lambda_{13}}\cos\left(\sqrt{\frac{\mu_{13}}{D_{13}}}x_3\right)\right\},\ x_3 \in [0,\, H_1], \tag{46}$$

which is called the Sturm-Liouville incomplete problem eigenfunction (39) corresponding to the eigenvalue μ_{13} (not yet found); A_{13} is an arbitrary constant. To find eigenvalue $\mu_{1,3}$, we should refer to conditions (18), (19), in which another function is involved – the desired function $c_2\left(x,\, t\right),\ \left(x,\, t\right) \in \Omega_{2x} \times \left[0,\, t_{END}\right]$ of auxiliary problem AP2. In other words, to find the eigenvalues and the corresponding eigenfunctions of the incomplete Sturm-Liouville problem (39), we will need to investigate the auxiliary problem AP2, which we will do in subsection 4.1 of this section.

Recall that in the course of studying the AP1 problem (still unfinished), we found out that all the eigenvalues of problems (37)-(39) of Sturm-Liouville are positive. Therefore, the constant μ_1 from (29) and (30), which is the sum of these eigenvalues, is also positive:

$$\begin{aligned} 0 &< \underbrace{\mu_{1nm}}_{\mu_1} = \mu_{11n} + \mu_{12m} + \mu_{13} \\ &= D_{11}\left(\frac{\alpha_{1n}}{L_1}\right)^2 + D_{12}\left(\frac{\beta_{1m}}{L_2}\right)^2 + \mu_{13},\ \ \forall n,\, m \in \mathbb{N}, \end{aligned} \tag{47}$$

where α_{1n} and β_{1m} are n-th and m-th positive roots of the transcendental equations (41) and (45), respectively; constituent constant $\mu_{13} > 0$, which is the eigenvalue of the incomplete Sturm-Liouville problem (39), has not yet been found (it means that the eigenfunction $X_{13}\left(x_3\right),\ x_3 \in \left[0,\, H_1\right]$, having a formal representation in the form (46) and corresponding to this eigenvalue, is not uniquely determined).

4.3 Partial investigation of the second auxiliary boundary-value problem

We will look for nontrivial solution of the AP2 problem in the following form

$$c_2\left(x,\, t\right) = T_2\left(t\right)\sum_{j=1}^{3} X_{2j}\left(x_j\right), \tag{48}$$

where meaning of requirements $X_{2j}\left(x_j\right) \not\equiv 0,\ j = \overline{1,\, 3}$ and $T_2\left(t\right) \not\equiv 0$ is obvious.

By analogy with the previous subsection 4.2, given the representation (48) in equation (20), we obtain

$$T_2'\left(t\right) + \mu_2 T_2\left(t\right) = 0, \tag{49}$$

$$\sum_{j=1}^{3} D_{2j} \cdot \frac{X''_{2j}\left(x_j\right)}{X_{2j}\left(x_j\right)} + \mu_2 = 0, \tag{50}$$

where μ_2 is the same constant μ_1 as in equations (39), (30), i.e. $\mu_2 = \mu_1$ (for convenience, we will use the notation μ_2 knowing that $\mu_2 = \mu_1$).

Remark 5. *The authors of this work know from experience, an inexperienced reader can easily erroneously assume that the constant* μ_2 *in equations* (49) *and* (50) *differs from the constant* μ_1 *in equations* (29) *and* (30)*: his erroneous assumption is facilitated by the fact that equations* (11) *and* (20) *are different equations acting in different layers with different physical properties (to be more exact, with different material-structural properties). Since such an erroneous assumption will necessarily lead to completely erroneous results, we would like, just in case, to justify the fact that these constants are the same within the framework of this remark. For this, we note that one equation*

$$\frac{\partial c\left(x,\, t\right)}{\partial t} = \sum_{j=1}^{3} D_j\left(x\right)\frac{\partial^2 c\left(x,\, t\right)}{\partial x_j^2} + f\left(x,\, t\right),$$

replaces the two original equations of (1)*, where:*

$$D_j\left(x\right) = \begin{cases} D_{1j} \equiv const. & if \quad x \in \mathrm{int}\Omega_{1x}, \\ D_{2j} \equiv const. & if \quad x \in \mathrm{int}\Omega_{2x}, \end{cases}$$

$$c(x,\,t)=\begin{cases} c_1(x,\,t) & if \quad (x,\,t)\in \mathrm{int}\Omega_{1x}\times(0,\,t_{END}]\,,\\ c_2(x,\,t) & if \quad (x,\,t)\in \mathrm{int}\Omega_{2x}\times(0,\,t_{END}]\,,\end{cases}$$

$$f(x,\,t)=\begin{cases} f_1(x,\,t) & if \quad (x,\,t)\in \mathrm{int}\Omega_{1x}\times(0,\,t_{END}]\,,\\ f_2(x,\,t) & if \quad (x,\,t)\in \mathrm{int}\Omega_{2x}\times(0,\,t_{END}]\,,\end{cases}$$

Consequently, the homogeneous equations of both auxiliary problems AP1 and AP2 are also described by one equation:

$$\frac{\partial c(x,\,t)}{\partial t}=\sum_{j=1}^{3}D_j(x)\,\frac{\partial^2 c(x,\,t)}{\partial x_j^2}. \tag{51}$$

Next, we introduce representation

$$c(x,\,t)=X(x)\,T(t)\,, \tag{52}$$

which is also one notation of two different representations (27) *and* (48)*, where*

$$X(x)=\begin{cases} X_1(x)=\prod\limits_{j=1}^{3}X_{1j}(x_j) & if \quad x\in\Omega_{1x},\\ X_2(x)=\prod\limits_{j=1}^{3}X_{2j}(x_j) & if \quad x\in\Omega_{2x},\end{cases}$$

$$T(t)=\begin{cases} T_1(t) & if \quad (x,\,t)\in \Omega_{1x}\times(0,\,t_{END}]\,,\\ T_1(t) & if \quad (x,\,t)\in \Omega_{2x}\times(0,\,t_{END}]\,.\end{cases}$$

Substitution of representations (52) *into the equation* (51) *gives us the following equality:*

$$\frac{T'(t)}{T(t)}=\sum_{j=1}^{3}D_j(x)\,\frac{\frac{\partial^2 X(x)}{\partial x_j^2}}{X(x)}. \tag{53}$$

Since in (53) *the left part depends only on the variable* t*, and the right side depends only on* x*, we conclude that the left and right sides of* (53) *are equal to the same constant:*

$$\frac{T'(t)}{T(t)}=\sum_{j=1}^{3}D_j(x)\,\frac{\frac{\partial^2 X(x)}{\partial x_j^2}}{X(x)}=-\mu\equiv const. \tag{54}$$

From (54) *we obtain the equations*

$$T'(t)+\mu T(t)=0,$$

$$\sum_{j=1}^{3}D_j(x)\,\frac{\partial^2 X(x)}{\partial x_j^2}+\mu X(x)=0,$$

which in expanded form are the following

$$\begin{cases} T'_1(t)+\mu T_1(t)=0 & if \quad (x,\,t)\in \Omega_{1x}\times(0,\,t_{END}]\,,\\ T'_2(t)+\mu T_2(t)=0 & if \quad (x,\,t)\in \Omega_{2x}\times(0,\,t_{END}]\,,\end{cases}$$

$$\begin{cases} \sum\limits_{j=1}^{3}D_{1j}\frac{X''_{1j}(x_j)}{X_{1j}(x_j)}+\mu=0 & if \quad x\in\Omega_{1x},\\ \sum\limits_{j=1}^{3}D_{1j}\frac{X''_{2j}(x_j)}{X_{2j}(x_j)}+\mu=0 & if \quad x\in\Omega_{2x}.\end{cases}$$

Now it is obvious that the same constant μ *participates in equations* (29), (30), (49) *and* (50)*: in equations* (29) *and* (30)*, which are related to the auxiliary problem AP1, this constant is denoted by* μ_1 *(not only for convenience: index 1 in* μ_1 *indicates the number of the auxiliary problem, but also for a more essential goal, which will be clear immediately after the end of this remark), and in equations* (49) *and* (50)*, which are related to the auxiliary problem AP2, the same constant is denoted as* μ_2 *(also not only for convenience).*

So, after an important Remark 5, let us return to the study of the obtained equations (49) and (50). First we deal with equation (50), and then we return to equation (49). We already know that constants and in equations (30) and (50) coincide. However, we do not have the right to require that in three homogeneous equations of the same type

$$D_{21}X''_{21}\left(x_1\right)+\mu_{21}X_{21}\left(x_1\right)=0, \tag{55}$$

$$D_{22}X''_{22}\left(x_2\right)+\mu_{22}X_{22}\left(x_2\right)=0, \tag{56}$$

$$D_{23}X''_{23}\left(x_3\right)+\mu_{23}X_{23}\left(x_3\right)=0, \tag{57}$$

which directly follow from equation (50) (see the transition procedure from equation (30) to equations (31)-(33)), constants μ_{21}, μ_{22} and μ_{23}, whose sum gives μ_2 (i.e. $\mu_2=\sum\limits_{j=1}^{3}\mu_{2j}$) coincide with the previous constituent constants μ_{11}, μ_{12}, μ_{13} (values μ_{11} and μ_{12} are already determined, and value μ_{13} will be determined in this subsection). The reason for this circumstance (i.e. the fact that $\mu_1=\mu_2$, but $\mu_{1j}\not\equiv\mu_{2j}$, $j=\overline{1,\,3}$) is due to the fact that the coefficients D_{1j} $\left(j=\overline{1,\,3}\right)$ in equations (31)-(33)) differ from the corresponding coefficients D_{2j} $\left(j=\overline{1,\,3}\right)$ in equations (55)-(57)). In other words, in the equations (55)-(57) constants μ_{21}, μ_{22}, μ_{23}, where $\sum\limits_{j=1}^{3}\mu_{2j}=\mu_2$, are still unknown constants and need to be determined. Finally, we note that equations of the same type (55)-(57) are related only by the fact that $\mu_2=\sum\limits_{j=1}^{3}\mu_{2j}$.

Further, the substitution of representations (48) to the boundary conditions (22)-(26) gives us the following boundary conditions:

- for function $X_{21}\left(x_1\right)$ two boundary conditions:

$$\begin{cases} X'_{21}\left(0\right)=0, \\ D_{21}X'_{21}\left(L_1\right)+\lambda_{21}X_{21}\left(L_1\right)=0; \end{cases} \tag{58}$$

- for function $X_{22}\left(x_2\right)$ again two boundary conditions:

$$\begin{cases} X'_{22}\left(0\right)=0, \\ D_{22}X'_{22}\left(L_2\right)+\lambda_{22}X_{22}\left(L_2\right)=0; \end{cases} \tag{59}$$

- for function $X_{23}\left(x_3\right)$ one boundary condition:

$$D_{23}X'_{23}\left(L_3\right)+\lambda_{23}X_{23}\left(L_3\right)=0. \tag{60}$$

Consequently, the appropriate combination of equations (55)-(57) and boundary conditions (58)-(60) again give us the following three Sturm-Liouville problems, the first two of which are complete problems, and the third problem, just like problem (39) is not a complete problem:

$$\begin{cases} D_{21}X''_{21}\left(x_1\right)+\mu_{21}X_{21}\left(x_1\right)=0,\ x_1\in\left(0,\,L_1\right), \\ X'_{21}\left(0\right)=0, \\ D_{21}X'_{21}\left(L_1\right)+\lambda_{21}X_{21}\left(L_1\right)=0; \end{cases} \tag{61}$$

$$\begin{cases} D_{22}X''_{22}\left(x_2\right)+\mu_{22}X_{22}\left(x_2\right)=0,\ x_2\in\left(0,\,L_2\right), \\ X'_{22}\left(0\right)=0, \\ D_{22}X'_{22}\left(L_2\right)+\lambda_{22}X_{22}\left(L_2\right)=0; \end{cases} \tag{62}$$

$$\begin{cases} D_{23}X''_{23}(x_3) + \mu_{23}X_{23}(x_3) = 0, \; x_3 \in (H_1,\, L_3), \\ D_{23}X'_{23}(L_3) + \lambda_{23}X_{23}(L_3) = 0. \end{cases} \tag{63}$$

Almost completely following the reasoning from subsection B in the study of problems (37) and (38), with respect to complete problems (61) and (62) of Sturm-Liouville, we can assert the following statements without detailed derivation:

- The complete problem (61) of Sturm-Liouville has eigenvalues

$$\mu_{21k} = D_{21}\left(\frac{\alpha_{2k}}{L_1}\right)^2 > 0, \; k \in \mathbb{N}, \tag{64}$$

  to which the following eigenfunctions correspond

$$X_{21k}(x_1) = A_{21k}\cos\left(\sqrt{\frac{\mu_{21k}}{D_{21}}}x_1\right), \; x_1 \in [0,\, L_1], \tag{65}$$

  and each of them is determined with precision to an arbitrary constant A_{21k}. In (65) number α_{2k} is k-th ($k \in \mathbb{N}$) positive root of the transcendental equation

$$\alpha_2 tg(\alpha_2) = \frac{\lambda_{21}L_1}{D_{21}}. \tag{66}$$

- The complete problem (62) of Sturm-Liouville has eigenvalues

$$\mu_{22p} = D_{22}\left(\frac{\beta_{2p}}{L_2}\right)^2 > 0, \; p \in \mathbb{N}, \tag{67}$$

  to which the following eigenfunctions correspond

$$X_{22p}(x_2) = A_{22p}\cos\left(\sqrt{\frac{\mu_{22p}}{D_{22}}}x_2\right), \; \forall x_2 \in [0,\, L_2], \tag{68}$$

  and each of them is determined with precision to an arbitrary constant A_{22p}. In (68) number β_{2p} is p-th ($p \in \mathbb{N}$) positive root of the transcendental equation

$$\beta_2 tg(\beta_2) = \frac{\lambda_{22}L_2}{D_{22}}. \tag{69}$$

Now let us study the incomplete Sturm-Liouville problem (63). It is easy to check that when $\mu_{23} > 0$ (in the case of $\mu_{23} \le 0$ problem (63) has only trivial solution) the general solution of (63) is function

$$X_{23}(x_3) = A_{23}\left\{\sin\left(\sqrt{\frac{\mu_{23}}{D_{23}}}x_3\right) - \cos\left(\sqrt{\frac{\mu_{23}}{D_{23}}}x_3\right) tg\left(\sqrt{\frac{\mu_{23}}{D_{23}}}L_3 + \theta\right)\right\}, \; x_3 \in [H_1,\, L_3], \tag{70}$$

where A_{23} is an arbitrary constant; $\theta = arctg\left(\frac{\sqrt{\mu_{23}D_{23}}}{\lambda_{23}}\right)$.

Remark 6. *The derivation of formula* (70) *is not difficult, but it requires a lot of calculations. In order to relatively easily convince the reader of validity of formula* (70)*, we propose a way of checking the answer, i.e. substitute formula* (70) *into* (63) *and use the following formulas, the validity of which is easily established:*

$$X'_{23}(x_3) = A_{23}\sqrt{\frac{\mu_{23}}{D_{23}}}\left\{\cos\left(\sqrt{\frac{\mu_{23}}{D_{23}}}x_3\right) + \sin\left(\sqrt{\frac{\mu_{23}}{D_{23}}}x_3\right) tg\left(\sqrt{\frac{\mu_{23}}{D_{23}}}L_3 + \theta\right)\right\},$$

$$X''_{23}(x_3) = A_{23}\frac{\mu_{23}}{D_{23}}\left\{\cos\left(\sqrt{\frac{\mu_{23}}{D_{23}}}x_3\right) tg\left(\sqrt{\frac{\mu_{23}}{D_{23}}}L_3 + \theta\right) - \sin\left(\sqrt{\frac{\mu_{23}}{D_{23}}}x_3\right)\right\},$$

$$tg\left(\sqrt{\frac{\mu_{23}}{D_{23}}}L_3 + \theta\right) = \frac{\lambda_{23}\sin\left(\sqrt{\frac{\mu_{23}}{D_{23}}}L_3\right) + \sqrt{\mu_{23}D_{23}}\cos\left(\sqrt{\frac{\mu_{23}}{D_{23}}}L_3\right)}{\lambda_{23}\cos\left(\sqrt{\frac{\mu_{23}}{D_{23}}}L_3\right) + \sqrt{\mu_{23}D_{23}}\sin\left(\sqrt{\frac{\mu_{23}}{D_{23}}}L_3\right)}.$$

Remark 7. *It is easy to verify that if in the formula* (70) *instead of the existing constant* A_{23} *we take* $A_{23}\left(\frac{\sqrt{\mu_{23}D_{23}}}{\lambda_{23}}\right)\sin\left(\sqrt{\frac{\mu_{23}}{D_{23}}}L_3\right)-\cos\left(\sqrt{\frac{\mu_{23}}{D_{23}}}L_3\right)$ *(this is valid due to the arbitrariness of the constant), then formula* (70) *will have the following form, similar to formula* (46) *for the function* $X_{13}(x_3)$, $x_3\in[0,\,H_1]$:

$$X_{23}(x_3)=A_{23}\left\{\sin\left(\sqrt{\frac{\mu_{23}}{D_{23}}}(L_3-x_3)\right)+\frac{\sqrt{\mu_{23}D_{23}}}{\lambda_{23}}\cos\left(\sqrt{\frac{\mu_{23}}{D_{23}}}(L_3-x_3)\right)\right\},\ x_3\in[H_1,\,L_3]. \tag{71}$$

Direct verification can easily show that the function $X_{23}(x_3)$, *defined by formula (71), satisfies the problem* (63). *In the future, we will use notation* (70).

Function $X_{23}(x_3)$, defined by formula (70), is called the eigenfunction of the incomplete Sturm-Liouville problem (63) corresponding to the eigenvalue μ_{23}.

Recall that in the course of studying the AP2 problem (still unfinished), we found out that all the eigenvalues of problems (61)-(63) of Sturm-Liouville are positive. Therefore, the constant μ_2 from (49) and (50), which is the sum of these eigenvalues, is also positive:

$$\begin{aligned}0<\underbrace{\mu_{2kp}}_{\mu_2}&=\mu_{21k}+\mu_{22p}+\mu_{23}\\&=D_{21}\left(\frac{\alpha_{2k}}{L_1}\right)^2+D_{22}\left(\frac{\beta_{2p}}{L_2}\right)^2+\mu_{23},\ \ \forall k,\,p\in\mathbb{N},\end{aligned} \tag{72}$$

where α_{2k} and β_{2p} are k-th ($k\in\mathbb{N}$) and p-th ($p\in\mathbb{N}$) positive roots of the transcendental equations (66) and (69), respectively; constituent constant $\mu_{23}>0$, which is eigenvalue of incomplete Sturm-Liouville problem (63), is still unknown (it means that the eigenfunction $X_{23}(x_3)$, $x_3\in[H_1,\,L_3]$, having a formal representation in the form (70) or (71) and corresponding to this eigenvalue, is not uniquely determined).

So, within the framework of the study of auxiliary problems AP1 and AP2, by this time the eigenvalues μ_{13} and μ_{23}, remain uncertain and, therefore, same thing can be said about their corresponding eigenfunctions $X_{13}(x_3)$, $x_3\in[0,\,H_1]$ and $X_{23}(x_3)$, $x_3\in[H_1,\,L_3]$; in addition, it is necessary to clarify the choice of constants A_{11n} $(n\in\mathbb{N})$, A_{12m} $(m\in\mathbb{N})$, A_{13}, A_{21k} $(k\in\mathbb{N})$, A_{22p} $(p\in\mathbb{N})$, A_{23}; finally, it is required to find functions $T_1(t)$ and $T_2(t)$, which satisfy equations (29) and (49).

4.4 Using the matching conditions, and the complete solving the both auxiliary problems

Recall that in the course of studying the auxiliary problems AP1 and AP2, we did not use matching conditions (18) and (19), and now it is time to use these conditions to find eigenvalues μ_{13} and μ_{23}, and redefine the corresponding eigenfunctions $X_{13}(x_3)$, $x_3\in[0,\,H_1]$ and $X_{23}(x_3)$, $x_3\in[H_1,\,L_3]$, formally represented by formulas (46) and (70) (or (71)), respectively. For this purpose, we first note that if equality

$$\underbrace{T_1(t)\prod_{j=1}^{3}X_{1j}(x_j)}_{c_1(x,t)}=\underbrace{T_1(t)\prod_{j=1}^{3}X_{2j}(x_j)}_{c_2(x,t)},$$

which is equivalents to

$$\frac{T_1(t)\prod\limits_{j=1}^{2}X_{1j}(x_j)}{T_1(t)\prod\limits_{j=1}^{2}X_{2j}(x_j)}=\frac{X_{23}(x_3)}{X_{13}(x_3)},$$

holds for $\forall x_j \in [0, L_j]$, $j = 1, 2$, $\forall t \in [0, T_{END}]$ and $\forall x_3 \in (H_1 - \varepsilon, H_1 + \varepsilon)$, $0 < \varepsilon \ll 1$, then it means that $X_{13}(x_3) = CX_{23}(x_3)$, $0 < \forall\varepsilon \ll 1$, where $C \neq 0$ is an arbitrary constant, which for convenience we choose as $\frac{A_{13}}{A_{23}}$, i.e. $C = \frac{A_{13}}{A_{23}}$ (such a choice of a constant is legitimate because of its arbitrariness, and, moreover, nothing will change from such (or other) choice). Now, having this fact, substituting representations (27) and (48) to matching conditions (18) and (19), we obtain

$$A_{23}X_{13}(x_3)\big|_{x_3=H_1^-} = A_{13}X_{23}(x_3)_{x_3=H_1^+},$$

$$A_{23}D_{13}X'_{13}(x_3)\big|_{x_3=H_1^-} = A_{13}D_{23}X'_{13}(x_3)\big|_{x_3=H_1^+}.$$

Taking into account formulas (46) and (70) in these two equalities, after performing the necessary calculations and transformations, gives us the following results: the desired eigenvalue μ_{23} from the Sturm-Liouville problem (63) are found by the formula

$$\mu_{23} = D_{23}\left(\frac{\gamma}{H_2}\right)^2 > 0, \tag{73}$$

and then the desired eigenvalue μ_{13} from the Sturm-Liouville problem (39) is calculated by the formula

$$\mu_{13} = \mu_{21} + \mu_{22} + \mu_{23} - \mu_{11} - \mu_{12}, \tag{74}$$

whose right side contains already found eigenvalues of Sturm-Liouville problems (37), (38), (61), (62).

In the formula (73) parameter γ is positive root of the transcendental equation

$$\frac{\sqrt{D_{23}\gamma^2 + \xi}}{\gamma} tg\left(\gamma + arctg\left(\frac{D_{23}\gamma}{H_2\lambda_{23}}\right)\right) = g(\gamma), \tag{75}$$

where

$$g(\gamma) = \frac{D_{23}}{\sqrt{D_{13}}} \frac{\sqrt{D_{13}(D_{23}\gamma^2 + \xi)} + H_2\lambda_{13} tg\left(\frac{H_1}{H_2}\sqrt{\frac{D_{23}\gamma^2+\xi}{D_{13}}}\right)}{\sqrt{D_{13}(D_{23}\gamma^2 + \xi)} tg\left(\frac{H_1}{H_2}\sqrt{\frac{D_{23}\gamma^2+\xi}{D_{13}}}\right) - H_2\lambda_{13}},$$

$$\xi = H_2^2(\mu_{21} + \mu_{22} - \mu_{11} - \mu_{12}).$$

Remark 8. *This note contains two important facts. The first important fact is that the parameter present in the transcendental equation (this parameter enters both the left side and the right side of the transcendental equation) depends on the eigenvalues of Sturm- Liouville problems* (37), (38), (61), (62). *Since each of these four Sturm-Liouville problems has an infinite number of eigenvalues (see formulas* (42), (44), (64), (68)*), it would be more correct to speak not about one transcendental equation* (75), *but about a family of transcendental equations. Moreover, having* $\{\alpha_{1n}\}_{n\in\mathbb{N}}$, *where* $0 < \alpha_{11} \leq \alpha_{12} \leq \ldots \leq \alpha_{1n} \leq \ldots$; $\{\beta_{1m}\}_{m\in\mathbb{N}}$, *where* $0 < \beta_{11} \leq \beta_{12} \leq \ldots \leq \beta_{1m} \leq \ldots$; $\{\alpha_{2k}\}_{k\in\mathbb{N}}$, *where* $0 < \alpha_{21} \leq \alpha_{22} \leq \ldots \leq \alpha_{2k} \leq \ldots$; $\{\beta_{2p}\}_{p\in\mathbb{N}}$, *where* $0 < \beta_{21} \leq \beta_{22} \leq \ldots \leq \beta_{2p} \leq \ldots$, *value of parameter* ξ *depends on a specific values if an ordered quadruple* $(n \in \mathbb{N}, m \in \mathbb{N}, k \in \mathbb{N}, p \in \mathbb{N})$, *where, generally speaking, order is important: for example, the quadruple* $(1, 1, 1, 2)$ *means that eigenvalue* μ_{11} *is calculated using the first positive root* (α_{11}) *of the equation* (41), *eigenvalue* μ_{12} *is calculated using the first positive root* (β_{11}) *of the equation* (45), *eigenvalue* μ_{21} *is calculated using the first positive root* (α_{21}) *of the equation* (66), *eigenvalue* μ_{21} *is calculated using the first positive root* (β_{21}) *of the equation* (69), *and, therefore,* $\xi = \xi(1, 1, 1, 2) = H_2^2(\mu_{211} + \mu_{222} - \mu_{111} - \mu_{121})$ *does not have to match the value* ξ, *found by quadruple* $(1, 1, 2, 1) : \xi(1, 1, 2, 1) = H_2^2(\mu_{212} + \mu_{221} - \mu_{111} - \mu_{121})$, *i.e. generally speaking,* $\xi(1, 1, 1, 2) \neq \xi(1, 1, 2, 1)$. *Finally, recall that positive roots* $\left\{\gamma_{q=q(n\in\mathbb{N}, m\in\mathbb{N}, k\in\mathbb{N}, p\in\mathbb{N})}\right\}_{q\in\mathbb{N}}$ *of family of equations* (75), *like the positive roots of all transcendental equations in this paper, are* *arranged in non-decreasing order:* $0 < \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_q \leq \ldots$, *where* $\gamma_1 = \min\limits_{n,\,m,\,k,\,p\in\mathbb{N}} \gamma_{q(n,\,m,\,k,\,p)}$, $\gamma_2 = \min\limits_{n,\,m,\,k,\,p\in\mathbb{N};\,q\neq 1} \{\gamma_q\}$, $\gamma_3 = \min\limits_{n,\,m,\,k,\,p\in\mathbb{N};\,q\neq 1,\,2} \{\gamma_q\}$, *etc.*

The second important fact of this remark is that despite the possible negative values of the parameter ξ, *all radicands in* (75) *are positive. Indeed, from* (73) *we have* $\gamma^2 = \frac{\mu_{23}}{D_{23}} H_2^2$, *and taking this expression into account in the expression* $D_{23}\gamma^2 + \xi$, *which causes the fear of the negativity of the radicands, we get:*

$$\begin{aligned} & D_{23}\gamma^2 + \xi = \mu_{23} H_2^2 + H_2^2 \left(\mu_{21} + \mu_{22} - \mu_{11} - \mu_{12}\right) \\ & = H_2^2 \left(\mu_{21} + \mu_{22} + \mu_{23} - \mu_{11} - \mu_{12}\right) \overset{(74)}{=} H_2^2 \mu_{13} > 0. \end{aligned}$$

.

Since the transcendental equation (75) has an infinite number of solutions, we arrive at the following results:

- The Sturm-Liouville problem (63) has an infinite number of eigenvalues

$$\mu_{23q} = D_{23} \left(\frac{\gamma_q}{H_2}\right)^2, \; q = q\,(n,\, m,\, k,\, p) \in, \; \forall n,\, m,\, k,\, p \in \mathbb{N}, \tag{76}$$

  to which the following eigenfunctions correspond

$$\begin{aligned} & X_{23q}\,(x_3) = \sin\left(\sqrt{\tfrac{\mu_{23q}}{D_{23}}} x_3\right) - tg\left(\sqrt{\tfrac{\mu_{23q}}{D_{23}}} L_3 + arctg\left(\frac{\sqrt{\mu_{23q} D_{23}}}{\lambda_{23}}\right)\right) \cos\left(\sqrt{\tfrac{\mu_{23q}}{D_{23}}} x_3\right), \\ & \forall x_3 \in [H_1,\, L_3]\,; \end{aligned} \tag{77}$$

  in (76) number $\gamma_q = \gamma_{q(n,\,m,\,k,\,p)}$ is q-th ($q \in \mathbb{N}$) positive root of the transcendental equation (75).

- The Sturm-Liouville problem (39) has an infinite number of eigenvalues

$$\begin{aligned} & \mu_{13nmkpq} = \mu_{21k} + \mu_{22p} + \mu_{23q} - \mu_{11n} - \mu_{12m}, \\ & q = q\,(n,\, m,\, k,\, p)\,, \;\; \forall n,\, m,\, k,\, p \in, \end{aligned}$$

  to which the following eigenfunctions correspond

$$X_{13nmkpq}\,(x_3) = \frac{\sqrt{\mu_{13nmkpq} D_{13}}}{\lambda_{13}} \cos\left(\sqrt{\frac{\mu_{13nmkpq}}{D_{13}}} x_3\right) + \sin\left(\sqrt{\frac{\mu_{13nmkpq}}{D_{13}}} x_3\right), \;\; \forall x_3 \in [0,\, H_1]\,. \tag{78}$$

So, within the framework of the study of auxiliary problems AP1 and AP2, two sub-problems remain unfinished: the problem of finding functions $T_1\,(t)$ and $T_2\,(t)$, for whose solution one, first of all, needs to clarify/redefine formulas (47) and (72) for constants μ_1 and μ_2; problem of choosing constants A_{11n} $(n \in \mathbb{N})$, A_{12m} $(m \in \mathbb{N})$, A_{21k} $(k \in \mathbb{N})$ and A_{22p} $(p \in \mathbb{N})$, which appear in formulas (43), (44), (65) and (68), respectively. We start by clarifying/redefining formulas (47) and (72) for constants μ_1 and μ_2. As it was noted in Remark 5, constants μ_1 and μ_2 coincide. Therefore, it suffices to clarify only the formula (47) for the constant μ_1 :

$$\begin{aligned} & 0 < \underbrace{\mu_{1nmkpq}}_{\mu_1 = \mu_2 \equiv \mu} = D_{21} \left(\tfrac{\alpha_{2k}}{L_1}\right)^2 + D_{22} \left(\tfrac{\beta_{2p}}{L_2}\right)^2 \\ & + D_{23} \left(\tfrac{\gamma_q}{H_2}\right)^2 - D_{11} \left(\tfrac{\alpha_{1n}}{L_1}\right)^2 - D_{12} \left(\tfrac{\beta_{1m}}{L_2}\right)^2. \end{aligned} \tag{79}$$

Now we clarify the problem of choosing constants A_{11n} $(n \in \mathbb{N})$, A_{12m} $(m \in \mathbb{N})$, A_{21k} $(k \in \mathbb{N})$ and A_{22p} $(p \in \mathbb{N})$, which appear in formulas (43), (44), (65) and (68), respectively. For this we use the fact that the system of eigenfunctions $\{y_n(x)\}_{n\in}$ of the Sturm-Liouville problem

$$\left\{\begin{array}{l}(p(x)\,y'(x))' - q(x)\,y(x) + \mu\rho(x)\,y(x) = 0,\ \forall x\in(a,\,b)\,,\\ \omega_{11}y'(a) + \omega_{12}y(a) = 0,\\ \omega_{21}y'(b) + \omega_{22}y(b) = 0,\end{array}\right. \tag{80}$$

where $\sum\limits_{j=1}^{2}\omega_{ij}^2 \neq 0,\ i=1,\,2;\ p(x)>0,\ x\in(a,\,b)\,;\ q(x)\geq 0,\ x\in(a,\,b)\,;\ \rho(x)>0,\ x\in(a,\,b)\,,$ forms an orthogonal system with weight $\rho(x)$ on the segment $[a,\,b]$, i.e.

$$\int\limits_a^b \rho(x)\,y_n(x)\,y_m(x)\,dx = 0$$

for $n\neq m$ (Al-Gwaiz, 2008; Levitan & Sargsyan, 1991).

Since our problems (37), (38), (61), (62) are problems of the form (80) ($q(x)\equiv 0$ and $\rho(x)\equiv 1$ for all four problems), we can state that the system of functions $\{X_{11n}(x_1)\}_{n\in\mathbb{N}}$, $\{X_{12m}(x_2)\}_{m\in\mathbb{N}}$, $\{X_{21k}(x_1)\}_{k\in\mathbb{N}}$, $\{X_{22p}(x_2)\}_{p\in\mathbb{N}}$, represented by formulas (43), (44), (65), (68), respectively, are orthogonal systems on segments $[0,\,L_1]$, $[0,\,L_2]$, $[0,\,L_1]$, $[0,\,L_2]$, respectively. One of the reasonable constraints on choice of constants $A_{11n}\ (n\in\mathbb{N})$, $A_{12m}\ (m\in\mathbb{N})$, $A_{21k}\ (k\in\mathbb{N})$, $A_{22p}\ (p\in\mathbb{N})$, which appear in formulas (43), (44), (65), (68), respectively, is the requirement of orthonormality of systems $\{X_{11n}(x_1)\}_{n\in\mathbb{N}}$, $\{X_{12m}(x_2)\}_{m\in\mathbb{N}}$, $\{X_{21k}(x_1)\}_{k\in\mathbb{N}}$, $\{X_{22p}(x_2)\}_{p\in\mathbb{N}}$ of eigenfunctions, i.e. requirement of satisfaction of conditions

$$\int\limits_a^b \rho(x)\,y_n(x)\,y_m(x)\,dx = \left\{\begin{array}{lll}0 & if & n\neq m\ ,\\ 1 & if & n=m\ .\end{array}\right.$$

Another option (very simple, but less reasonable) of choosing constants is simply equating them to some number, for example, to 1. In this paper, we choose the first option:

- From the requirement of satisfaction of condition $\int\limits_0^{L_1} X_{11n}^2(x_1)\,dx_1 = 1$ it follows that $A_{11n} = \sqrt{\frac{\lambda_{11}}{D_{11}} + \frac{\mu_{11n}}{\lambda_{11}}},\ \forall n\in\mathbb{N};$
- From the requirement of satisfaction of condition $\int\limits_0^{L_2} X_{12m}^2(x_2)\,dx_2 = 1$ it follows that $A_{12m} = \sqrt{\frac{\lambda_{12}}{D_{12}} + \frac{\mu_{12m}}{\lambda_{12}}},\ \forall m\in\mathbb{N};$
- From the requirement of satisfaction of condition $\int\limits_0^{L_1} X_{21k}^2(x_1)\,dx_1 = 1$ it follows that $A_{21k} = \sqrt{\frac{\lambda_{21}}{D_{21}} + \frac{\mu_{21k}}{\lambda_{21}}},\ \forall k\in\mathbb{N};$
- From the requirement of satisfaction of condition $\int\limits_0^{L_2} X_{22p}^2(x_2)\,dx_2 = 1$ it follows that $A_{22p} = \sqrt{\frac{\lambda_{22}}{D_{22}} + \frac{\mu_{22p}}{\lambda_{22}}},\ \forall p\in\mathbb{N}.$

It should be noted here that we did not ensure that the orthogonal systems of eigenfunctions $\{X_{13nmkpq}(x)\}_{n,\,m,\,k,\,p,\,q\in\mathbb{N}}$ and $\{X_{23nmkpq}(x)\}_{n,\,m,\,k,\,p,\,q\in\mathbb{N}}$, defined by the already established formulas (78) and (77), respectively, became orthonormal systems: this was not necessary in this work, although it would not be difficult to ensure.

So, the final formulas for calculating the eigenfunctions $X_{11n}(x_1)$, $X_{12m}(x_2)$, $X_{21k}(x_1)$, $X_{22p}(x_2)$ are the following formulas:

$$X_{11n}(x_1) = \sqrt{\frac{\lambda_{11}}{D_{11}} + \frac{\mu_{11n}}{\lambda_{11}}} \sin\left(\sqrt{\frac{\mu_{11n}}{D_{11}}} x_1\right), \ \forall x_1 \in [0,\, L_1], \ \forall n \in \mathbb{N}; \tag{81}$$

$$X_{12m}(x_2) = \sqrt{\frac{\lambda_{12}}{D_{12}} + \frac{\mu_{12m}}{\lambda_{12}}} \sin\left(\sqrt{\frac{\mu_{12m}}{D_{12}}} x_2\right), \ \forall x_2 \in [0,\, L_2], \ \forall m \in \mathbb{N}; \tag{82}$$

$$X_{21k}(x_1) = \sqrt{\frac{\lambda_{21}}{D_{21}} + \frac{\mu_{21k}}{\lambda_{21}}} \sin\left(\sqrt{\frac{\mu_{21k}}{D_{21}}} x_1\right), \ \forall x_1 \in [0,\, L_1], \ \forall k \in \mathbb{N}; \tag{83}$$

$$X_{22p}(x_1) = \sqrt{\frac{\lambda_{22}}{D_{22}} + \frac{\mu_{22p}}{\lambda_{22}}} \sin\left(\sqrt{\frac{\mu_{22p}}{D_{22}}} x_2\right), \ \forall x_2 \in [0,\, L_2], \ \forall p \in \mathbb{N}. \tag{84}$$

Now we can proceed to solving the last problem in the framework of the study of auxiliary problems AP1 and AP2 - the problem of finding functions $T_1(t)$ and $T_2(t)$ from equations (29) and (49), respectively. Taking into account Remark 5, equations (29) and (49) completely coincide: $T_1(t) = T_2(t) = T(t)$; however, due to conditions (12) and (21), the function $T(t)$ must be different on layers Ω_{1x} and Ω_{2x}, i.e.

$$T(t) = \begin{cases} T_1(t) & if \quad (x,\, t) \in \Omega_{1x} \times (0,\, t_{END}], \\ T_1(t) & if \quad (x,\, t) \in \Omega_{2x} \times (0,\, t_{END}]. \end{cases}$$

As was emphasized in Remark 2, this circumstance is highly atypical and it introduces some peculiarity to the investigated problem.

If we consider equations (29) and (49) only from the position of time $t \in [0,\, t_{END}]$, rather than from the position of spatial variables, the solution of these equations is the function $T(t) = Be^{-\mu \cdot t}$, where B is some coefficient that is not yet defined. Since number $\mu\,(= \mu_1 = \mu_2)$ is determined by finally found formula (79), we can write

$$T_{nmkpq}(t) = B_{nmkpq} e^{-\mu_{nmkpq} \cdot t}, \ \forall n,\, m,\, k,\, p,\, q \in \mathbb{N}, \tag{85}$$

where coefficients B_{nmkpq} are to be determined taking into account spatial variables.

To satisfy conditions (12) and (21), in the formula (85) for the layer Ω_{1x} there should be its own distinctive coefficients B_{1nmkpq}, and for the layer Ω_{2x} there should be its own distinctive constants B_{2nmkpq}, i.e.

$$T_{nmkpq}(t) = \begin{cases} B_{1nmkpq} e^{-\mu_{nmkpq} \cdot t} & if \quad x \in \Omega_{1x}, \\ B_{2nmkpq} e^{-\mu_{nmkpq} \cdot t} & if \quad x \in \Omega_{2x} \end{cases} \tag{86}$$

for $\forall t \in [0,\, t_{END}]$ and $\forall n,\, m,\, k,\, p,\, q \in \mathbb{N}$.

Taking into account formula (81) for $X_{11n}(x_1)$, formula (82) for $X_{12m}(x_2)$, formula (78) for $X_{13nmkpq}(x_3)$, formula (83) for $X_{21k}(x_1)$, formula (84) for $X_{22p}(x_2)$, formula (77) for $X_{23nmkpq}(x_3)$, formula (86) for $T_{nmkpq}(t)$ in representations (27) and (48), we obtain the following formula for the desired functions $c_1(x,\, t)$, where $(x,\, t) \in \Omega_{1x} \times [0,\, t_{END}]$, and $c_2(x,\, t)$, where $(x,\, t) \in \Omega_{2x} \times [0,\, t_{END}]$:

$$c_1(x,\, t) = \sum_{n,\, m,\, k,\, p,\, q=1}^{+\infty} B_{1nmkpq} e^{-\mu_{nmkpq} \cdot t} X_{1nmkpq}(x), \tag{87}$$

where $X_{1nmkpq}(x) = X_{11n}(x_1)\, X_{12m}(x_2)\, X_{13nmkpq}(x_3)$, $x_1 \in [0,\, L_1]$, $x_2 \in [0,\, L_2]$, $x_3 \in [0,\, H_1]$, $t \in [0,\, t_{END}]$, system of functions $\{X_{1nmkpq}(x)\}_{n,\, m,\, k,\, p,\, q \in \mathbb{N}}$ is an orthogonal system, i.e.

$$\int\limits_{\Omega_{1x}} X_{1nmkpq}(x)\, X_{1NMKPQ}(x)\, dx = 0 \tag{88}$$

if $n \neq N,\, m \neq M,\, k \neq K,\, p \neq P,\, q \neq Q$;

$$c_2(x,\,t) = \sum_{n,\,m,\,k,\,p,\,q=1}^{+\infty} B_{2nmkpq} e^{-\mu_{nmkpq}\cdot t} X_{2nmkpq}(x), \tag{89}$$

where $X_{2nmkpq}(x) = X_{21k}(x_1)\, X_{22p}(x_2)\, X_{23nmkpq}(x_3)\,,\, x_1 \in [0,\, L_1]\,,\, x_2 \in [0,\, L_2]\,,\, x_3 \in [H_1,\, L_3]\,,$ $t \in [0,\, t_{END}]\,,$ system of functions $\{X_{2nmkpq}(x)\}_{n,\,m,\,k,\,p,\,q \in \mathbb{N}}$ is an orthogonal system, i.e.

$$\int\limits_{\Omega_{2x}} X_{2nmkpq}(x)\, X_{2NMKPQ}(x)\, dx = 0 \tag{90}$$

if $n \neq N,\, m \neq M,\, k \neq K,\, p \neq P,\, q \neq Q$.

Obviously, the function $c_1(x,\,t)\,,$ determined by the formula (87), satisfies all homogeneous boundary conditions (13)-(17) of the auxiliary problem AP1, since they are satisfied by all members of the quadruple series in the right-hand side (87); similarly function $c_2(x,\,t)\,,$ determined by the formula (89), satisfies all homogeneous boundary conditions (22)-(26) of the auxiliary problem AP1, since they are satisfied by all members of the quadruple series in the right-hand side of (89); in addition, these functions satisfy the matching conditions (18), (19), since functions $X_{13nmkpq}(x_3)$ and $X_{23nmkpq}(x_3)\,,$ which are contained in each member of the quadruple series of (87) and (89), respectively, automatically satisfy the matching conditions (18), (19) – functions $X_{13nmkpq}(x_3)$ and $X_{23nmkpq}(x_3)$ were determined owing to the conditions (18), (19). Therefore, it remains to enforce functions $c_1(x,\,t)$ and $c_2(x,\,t)$ to satisfy the initial conditions (12) and (21), respectively.

The sought-for function $c_1(x,\,t)\,,$ which is determined by formula (87), to satisfy initial condition (12), we obtain:

$$c_{10}(x) = \sum_{n,\,m,\,k,\,p,\,q=1}^{+\infty} B_{1nmkpq} X_{1nmkpq}(x). \tag{91}$$

Analogously, the sought-for function $c_2(x,\,t)$ determined by formula (89) to satisfy the initial condition (21), we obtain:

$$c_{20}(x) = \sum_{n,\,m,\,k,\,p,\,q=1}^{+\infty} B_{2nmkpq} X_{2nmkpq}(x). \tag{92}$$

Let us by turns apply to (91) and (92) one of the fundamental theorems of mathematical physics - Steklov's Theorem on decomposability of any twice continuously differentiable function into absolutely and uniformly convergent series by orthogonal system of eigenfunctions of the Sturm-Liouville problem (first strictly proved in Steklov (1983); see also Levitan & Sargsyan (1991)). Let us start with (91).

After multiplying both parts of (91) by the function $X_{1NMKPQ}(x)$ and integrating the resulting equality over the layer Ω_{1x}, we will have

$$\int\limits_{\Omega_{1x}} c_{10}(x)\, X_{1NMKPQ}(x)\, dx = \sum_{n,\,m,\,k,\,p,\,q=1}^{+\infty} B_{1nmkpq} \int\limits_{\Omega_{1x}} X_{1nmkpq}(x)\, X_{1NMKPQ}(x)\, dx, \tag{93}$$

where the introduction of the integral sign under the sign of the series means termwise integration of functional series, and we have the right to do this, since according to (Il'in & Poznyak, 1980), first, for the layer Ω_{1x} the series (91) converges uniformly, and, secondly, we integrate each member $B_{1nmkpq} X_{1nmkpq}(x)$ of the series (91) for the layer Ω_{1x} (here we deliberately did not use a stronger fact – the fact of the continuity of each member of the series (91), since the requirement of integrability of each member of the series is sufficient, and there is no need to require that all members of the series were continuous, as it is supposed in various textbooks on mathematical analysis).

Since the left side of equality (93) is non-zero, then by virtue of (88) we can state that the right side of equality (93) contains only one non-zero term that is a $N \cdot M \cdot K \cdot P \cdot Q$-th member of the quadruple series, i.e. member of the series where $n = N$, $m = M$, $k = K$, $p = P$, $q = Q$. In other words we will have an equality

$$\int\limits_{\Omega_{1x}} c_{10}\,(x)\,X_{1NMKPQ}\,(x)\,dx = B_{1NMKPQ}\int\limits_{\Omega_{1x}} X^2_{1NMKPQ}\,(x)\,dx,$$

from which it immediately follows that for $\forall n,\ m,\ k,\ p,\ q \in \mathbb{N}$ there takes place

$$B_{1nmkpq} = \frac{\int\limits_{\Omega_{1x}} c_{10}\,(y)\,X_{1nmkpq}\,(y)\,dy}{\int\limits_{\Omega_{1z}} X^2_{1nmkpq}\,(z)\,dz} = \frac{\int\limits_0^{L_1} dy_1 \int\limits_0^{L_2} dy_2 \int\limits_0^{H_1} c_{10}\,(y)\,X_{1nmkpq}\,(y)\,dy_3}{\int\limits_0^{L_1} dz_1 \int\limits_0^{L_2} dz_2 \int\limits_0^{H_1} X^2_{1nmkpq}\,(z)\,dz_3}, \tag{94}$$

Now we will consider (92) and act in exactly the same way as we did when considering equality (91). Namely, multiplying both parts of (92) by the function $X_{2NMKPQ}\,(x)$ and integrating the obtained equality over the layer Ω_{2x}, we will have the equality

$$\int\limits_{\Omega_{1x}} c_{10}\,(x)\,X_{1NMKPQ}\,(x)\,dx = \sum_{n,\,m,\,k,\,p,\,q=1}^{+\infty} B_{1nmkpq}\int\limits_{\Omega_{1x}} X_{1nmkpq}\,(x)\,X_{1NMKPQ}\,(x)\,dx,$$

from which taking into account (90) we will get

$$\int\limits_{\Omega_{2x}} c_{20}\,(x)\,X_{2NMKPQ}\,(x)\,dx = B_{2NMKPQ}\int\limits_{\Omega_{1x}} X^2_{2NMKPQ}\,(x)\,dx,$$

i.e. we have obtained that for $\forall n,\ m,\ k,\ p,\ q \in \mathbb{N}$ there takes place

$$B_{2nmkpq} = \frac{\int\limits_{\Omega_{2y}} c_{20}\,(y)\,X_{2nmkpq}\,(y)\,dy}{\int\limits_{\Omega_{2z}} X^2_{2nmkpq}\,(z)\,dz} = \frac{\int\limits_0^{L_1} dy_1 \int\limits_0^{L_2} dy_2 \int\limits_{H_1}^{L_3} c_{20}\,(y)\,X_{2nmkpq}\,(y)\,dy_3}{\int\limits_0^{L_1} dz_1 \int\limits_0^{L_2} dz_2 \int\limits_{H_1}^{L_3} X^2_{2nmkpq}\,(z)\,dz_3}. \tag{95}$$

It is not difficult to prove (Tikhonov & Samarsky, 1990) that the function $c_1\,(x,\,t)$ defined by formulas (87), (94) is a continuously differentiable function by a variable t in the interval $[0,\,t_{END}]$ and twice continuously differentiable function by variable x for the layer Ω_{1x}, which satisfies the equation (11). Similarly, a function $c_2\,(x,\,t)$, defined by formulas (89), (95) is a continuously differentiable function for a variable in a segment and a twice continuously differentiable function (twice differentiable function) for a variable t for the layer Ω_{2x}, which satisfies equation (20). Thus, the functions $c_1\,(x,\,t)$ and $c_2\,(x,\,t)$ are continuous functions for $\Omega_{1x} \times [0,\,t_{END}]$ and $\Omega_{2x} \times [0,\,t_{END}]$, respectively, and since these functions satisfy the matching conditions (18), (19) (it has been proved above), they are considered to be solutions of auxiliary problems AP1 and AP2, respectively.

Thus, the study of auxiliary problems AP1 and AP2 is entirely completed, and now we can proceed to finding a solution for the original problem (1)-(8).

4.5 Solving the original problem (1)-(8)

Obviously, substituting (94) in (87) and (95) in (89), we get the following representations for the functions $c_1(x, t)$, $\forall (x, t) \in \Omega_{1x} \times [0, t_{END}]$ and $c_2(x, t)$, $\forall (x, t) \in \Omega_{2x} \times [0, t_{END}]$:

$$c_1(x, t) = \int\limits_{\Omega_{1y}} G_1(x, y, t)\, c_{10}(y)\, dy, \tag{96}$$

where

$$G_1(x, y, t) = \sum_{n, m, k, p, q=1}^{+\infty} e^{-\mu_{nmkpq} t} \frac{X_{1nmkpq}(x)\, X_{1nmkpq}(y)}{\int\limits_{\Omega_{1z}} X^2_{1nmkpq}(z)\, dz}; \tag{97}$$

$$c_2(x, t) = \int\limits_{\Omega_{2y}} G_2(x, y, t)\, c_{20}(y)\, dy, \tag{98}$$

where

$$G_2(x, y, t) = \sum_{n, m, k, p, q=1}^{+\infty} e^{-\mu_{nmkpq} \cdot t} \frac{X_{2nmkpq}(x)\, X_{2nmkpq}(y)}{\int\limits_{\Omega_{2z}} X^2_{2nmkpq}(z)\, dz}, \tag{99}$$

which are a more compact form for auxiliary problems AP1 and AP2 solutions, respectively. Each of the above introduced functions $G_j(x, y, t)$, $j = 1, 2$ is a well-known and deeply studied Green's function (Tikhonov & Samarsky, 1990; Levitan & Sargsyan, 1991; Abrikosov et al., 1963; Levitov & Shitov, 2002)

$$G(x, y, t) = \sum_{n_1 \ldots n_m = 1}^{+\infty} e^{-\mu_{n_1 \ldots n_m} \cdot t} \frac{X_{n_1 \ldots n_m}(x)\, X_{n_1 \ldots n_m}(y)}{\int\limits_{\Omega_z} X^2_{n_1 \ldots n_m}(z)\, dz}.$$

Our goal in this subsection is the analytical construction of the solution to the original problem (1)-(8), using the Green's functions $G_j(x, y, t)$, $j = 1, 2$. As we will be able to see below, after completing the study of auxiliary problems AP1, AP2, there is no difficulty in finding an analytical solution to the original problem (1)-(8): a more or less difficult part of the research for the problem considered in this paper is the study of auxiliary problems AP1 and AP2.

Let us formulate a new auxiliary problem, naming it NAP1: it is required to find solutions to the inhomogeneous equation (which coincides with equation (1) for $i = 1$)

$$\frac{\partial c_1(x, t)}{\partial t} = \sum_{j=1}^{3} D_{1j} \frac{\partial^2 c_1(x, t)}{\partial x_j^2} + f_1(x, t), \ (x, t) \in \mathrm{int}\Omega_{1x} \times (0, t_{END}], \tag{100}$$

which satisfies the zero-initial condition

$$\left. c_1(x, t) \right|_{t=0^+} = 0, \ x \in \Omega_{1x} \tag{101}$$

and zero-boundary conditions (13)-(17) of the auxiliary problem AP1.

We will search for solution to the NAP1 problem according to the orthogonal system of functions $\{X_{1nmkpq}(x)\}_{n, m, k, p, q \in \mathbb{N}}$ already constructed by us, i.e. in the form

$$c_1(x, t) = \sum_{n, m, k, p, q=1}^{+\infty} c_{1nmkpq}(t)\, X_{1nmkpq}(x), \tag{102}$$

where the functional coefficients $c_{1nmkpq}(t)$ are still unknown and to be determined.

The source function $f_1(x,\, t)$ from equation (100) will also be expanded in an orthogonal system of functions $\{X_{1nmkpq}(x)\}_{n,\, m,\, k,\, p,\, q\in\mathbb{N}}$, i.e.

$$f_1(x,\, t) = \sum_{n,\, m,\, k,\, p,\, q=1}^{+\infty} f_{1nmkpq}(t)\, X_{1nmkpq}(x), \tag{103}$$

where the coefficients $f_{1nmkpq}(t)$ are calculated by the following formula (this formula is easy to establish by analogy with the procedure for finding the coefficients (94) and (95)):

$$f_{1nmkpq}(t) = \frac{\int\limits_{\Omega_{1y}} f_1(y,\, t)\, X_{1nmkpq}(y)\, dy}{\int\limits_{\Omega_{1z}} X^2_{1nmkpq}(z)\, dz}. \tag{104}$$

By substituting expansions (102) and (103) into equation (100), we will get

$$\begin{gathered}\sum_{n,\, m,\, k,\, p,\, q=1}^{+\infty} X_{1nmkpq}(x) \left\{ c'_{1nmkpq}(t) - \left(D_{11}\frac{X''_{11n}(x_1)}{X_{11n}(x_1)} + D_{12}\frac{X''_{12m}(x_2)}{X_{12nm}(x_2)} \right.\right. \\ \left.\left. + D_{13}\frac{X''_{13nmkpq}(x_3)}{X_{13nmkpq}(x_3)} \right) c_{1nmkpq}(t) - f_{1nmkpq}(t) \right\} = 0.\end{gathered}$$

Since the resulting equality is nothing else than decomposition of the zero function in an orthogonal system $\{X_{1nmkpq}(x)\}_{n,\, m,\, k,\, p,\, q\in\mathbb{N}}$, and since the zero function can have only zero coefficients in the decomposition, we can state that for $\forall n,\, m,\, k,\, p,\, q \in \mathbb{N}$ there will take place

$$\begin{gathered} c'_{1nmkpq}(t) - \left(D_{11}\frac{X''_{11n}(x_1)}{X_{11n}(x_1)} + D_{12}\frac{X''_{12m}(x_2)}{X_{12nm}(x_2)} \right. \\ \left. + D_{13}\frac{X''_{13nmkpq}(x_3)}{X_{13nmkpq}(x_3)} \right) c_{1nmkpq}(t) - f_{1nmkpq}(t) = 0. \end{gathered} \tag{105}$$

Since the functions $X_{1n}(x_1)\,,\, X_{12nm}(x_2)\,,\, X_{13nmkpq}(x_3)$ satisfy equations (31)-(33), respectively, the following equality is valid:

$$\begin{gathered} D_{11}\frac{X''_{11n}(x_1)}{X_{11n}(x_1)} + D_{12}\frac{X''_{12m}(x_2)}{X_{12nm}(x_2)} + D_{13}\frac{X''_{13nmkpq}(x_3)}{X_{13nmkpq}(x_3)} \\ = -\left(\mu_{11n} + \mu_{12m} + \mu_{13nmkpq}\right) \overset{(74)}{=} \underbrace{-\mu_{nmkpq}}_{\mu_{1nmkpq}=\mu_{2nmkpq}}. \end{gathered}$$

Taking into account this fact in equation (105), we will obtain the following ordinary differential equation with constant coefficients for the desired functional expansion coefficients $c_{1nmkpq}(t)$ of the expansion (102):

$$c'_{1nmkpq}(t) + \mu_{nmkpq} c_{1nmkpq}(t) - f_{1nmkpq}(t) = 0. \tag{106}$$

For the unique solvability of equation (106), only one initial condition is required, and this condition is

$$c_{1nmkpq}(0) = 0,\ \forall n,\, m,\, k,\, p,\, q \in \mathbb{N}, \tag{107}$$

which follows from the requirement that the function $c_1(x,\, t)$, represented by the decomposition (102) satisfy the zero-initial condition (101):

$$0 = \sum_{n,\, m,\, k,\, p,\, q=1}^{+\infty} c_{1nmkpq}(0)\, X_{1nmkpq}(x),\ \forall x \in \Omega_{1x}.$$

It is easy to verify that the solution to problem (106), (107) is the function

$$c_{1nmkpq}(t) = \int\limits_0^t e^{-\mu_{nmkpq}(t-\tau)} f_{1nmkpq}(\tau)\, d\tau. \tag{108}$$

By substituting the expression (104) into (108), and then substituting the resulting expression into decomposition (102) and carrying out simple transformations, taking into account the fact of uniform convergence of the fourfold series (it means that we can swap the series sign and the integral sign) we will obtain

$$c_1(x, t) = \int_0^t d\tau \int_{\Omega_{1y}} G_1(x, y, t-\tau) f_1(y, \tau)\, dy, \tag{109}$$

where the function $G_1(x, y, t-\tau)$ is the same Green function (97), in which instead of the argument t there is an argument $t-\tau$.

Now let us formulate a new auxiliary problem NAP2: it is required to find solutions to the inhomogeneous equation (which coincides with equation (1) at $i=2$)

$$\frac{\partial c_2(x, t)}{\partial t} = \sum_{j=1}^{3} D_{2j} \frac{\partial^2 c_2(x, t)}{\partial x_j^2} + f_2(x, t), \ (x, t) \in \operatorname{int}\Omega_{2x} \times (0, t_{END}], \tag{110}$$

which satisfies the zero-initial condition

$$\left. c_2(x, t) \right|_{t=0^+} = 0, \ x \in \Omega_{2x}$$

and to zero-boundary conditions (22)-(26) of the auxiliary problem AP2.

Having carried out the necessary calculations in a similar way with the corresponding calculations in finding the solution (109) of the NAP1 problem, we obtain the desired solution of the NAP2 problem:

$$c_2(x, t) = \int_0^t d\tau \int_{\Omega_{2y}} G_2(x, y, t-\tau) f_2(y, \tau)\, dy, \tag{111}$$

where the function $G_2(x, y, t-\tau)$ is the same Green's function (99), in which instead of the argument t there is an argument $t-\tau$.

Let us summarize the intermediate results obtained by this time. Since the equations (1) of the original problem (1)-(8) are linear equations (like the corresponding equations of the problems AP1, AP2, NAP1, NAP2), then:

- the sum of the functions determined by formulas (96) and (109) is the solution of the problem consisting of the non-homogeneous equation (100), nonzero-initial condition (12) and zero-boundary conditions (13)-(17).
- the sum of the functions determined by formulas (98) and (111) is the solution of the problem consisting of the non-homogeneous equation (111), the nonzero-initial condition (21) and zero-boundary conditions (22)-(26).

In other words, functions

$$c_j(x, t) = \int_{\Omega_{jy}} G_j(x, y, t)\, c_{j0}(y)\, dy + \int_0^t d\tau \int_{\Omega_{jy}} G_j(x, y, t-\tau) f_j(y, \tau)\, dy, \ \ j = 1, 2 \tag{112}$$

give us a solution to problem (1)-(8) provided that $c_{j1}(x_2, x_3, t) \equiv 0$, $a_{j1}(x_2, x_3, t) \equiv 0$, $c_{j2}(x_1, x_3, t) \equiv 0$, $a_{j2}(x_1, x_3, t) \equiv 0$, $a_{j3}(x_1, x_2, t) \equiv 0$ for $\forall j = 1, 2$. Therefore, to complete our study, it remains to find a solution to problem (1)-(8), provided that $f_j(x, y, t) \equiv 0$ and $c_{j0}(x) \equiv 0$ for $\forall j = 1, 2$, and then, add the solution found to the right side of the formula (112). To achieve this, we will use the following properties of the Green's functions $G_j(x, y, t)$, $j = 1, 2$ (Levitov & Shitov, 2002), (Babich et al., 1964):

- first, functions $G_j(x,\, y,\, t)\,,\ j = 1,\, 2$ can be represented in a multiplicative form

$$G_j(x,\, y,\, t) = \prod_{i=1}^{3} G_{ji}(x_i,\, y_i,\, t),\ \forall j = 1,\, 2,$$

where

$$G_{ji}(x_i,\, y_i,\, t) = \sum_{\ell=1}^{+\infty} e^{-\mu_{ji\ell}\cdot t} \frac{X_{ji\ell}(x_i)\, X_{ji\ell}(y_i)}{\int\limits_{z1}^{z2} X_{ji\ell}^2(z)\, dz},$$

$$\begin{cases} if\ \ (j,\, i) = (1,\, 1) \ \Rightarrow\ (\ell,\, z1,\, z2) = (n,\, 0,\, L_1)\,, \\ if\ \ (j,\, i) = (1,\, 2) \ \Rightarrow\ (\ell,\, z1,\, z2) = (m,\, 0,\, L_2)\,, \\ if\ \ (j,\, i) = (1,\, 3) \ \Rightarrow\ (\ell,\, z1,\, z2) = (q,\, 0,\, H_1)\,, \\ if\ \ (j,\, i) = (2,\, 1) \ \Rightarrow\ (\ell,\, z1,\, z2) = (k,\, 0,\, L_1)\,, \\ if\ \ (j,\, i) = (2,\, 2) \ \Rightarrow\ (\ell,\, z1,\, z2) = (p,\, 0,\, L_2)\,, \\ if\ \ (j,\, i) = (2,\, 3) \ \Rightarrow\ (\ell,\, z1,\, z2) = (q,\, H_1,\, L_3)\,, \\ q = q(n,\, m,\, k,\, p)\,; \end{cases}$$

- second, for $\forall j = 1,\, 2$ the Green's function $G_j(x,\, y,\, t)$ is the solution of the initial-boundary problem

$$\frac{\partial G_j(x,\, y,\, t)}{\partial t} = \sum_{j=1}^{3} D_{2j} \frac{\partial^2 G_j(x,\, y,\, t)}{\partial x_j^2},$$

$$G_j(x,\, y,\, t)\big|_{t=0^+} = \prod_{i=1}^{3} \delta(x_i - y_i),$$

$$\left.\frac{\partial G_j(x,\, y,\, t)}{\partial x_1}\right|_{x_1=0^+} = 0,$$

$$\left.\left[D_{ij} \frac{\partial G_j(x,\, y,\, t)}{\partial x_1} + \lambda_{j1} G_j(x,\, y,\, t)\right]\right|_{x_1=L_1^-} = 0,$$

$$\left.\frac{\partial G_j(x,\, y,\, t)}{\partial x_2}\right|_{x_2=0^+} = 0,$$

$$\left.\left[D_{j2} \frac{\partial G_j(x,\, y,\, t)}{\partial x_2} + \lambda_{j2} G_j(x,\, y,\, t)\right]\right|_{x_2=L_2^-} = 0,$$

$$\left.\left[D_{j3} \frac{\partial G_j(x,\, y,\, t)}{\partial x_3} + (2j-3)\, \lambda_{j3} G_j(x,\, y,\, t)\right]\right|_{x_3=(j-1)^+ L_3^-} = 0,$$

where at $j = 1$ problem area is a layer Ω_{1x}, at $j = 2$ problem area is a layer Ω_{2x}, function $\delta(x_i - y_i)$ means the Dirac delta function; the variable $y = (y_1,\, y_2,\, y_3)$ participates in the problem as a parameter, at that $y \in \Omega_{1y}$ when $j = 1$ and $y \in \Omega_{2y}$ when $j = 2$.

By using the above-listed properties of the Green's function $G_j(x,\, y,\, t)\,,\ j = 1,\, 2$ we can write the following formulas, the validity of which can be easily proved by direct verification:

- function $c_1(x, t)$, which is defined as

$$\begin{gathered}
c_1(x, t) = \int_0^t d\tau \int_0^{L_2} dy_2 \int_0^{H_1} G_1(x, y, t-\tau)\big|_{y_1=0} \frac{-c_{11}(y_2, y_3, \tau)}{D_{11}} dy_3 \\
+ \int_0^t d\tau \int_0^{L_2} dy_2 \int_0^{H_1} G_1(x, y, t-\tau)\big|_{y_1=L_1} a_{11}(y_2, y_3, \tau)\, dy_3 \\
+ \int_0^t d\tau \int_0^{L_1} dy_1 \int_0^{H_1} G_1(x, y, t-\tau)\big|_{y_2=0} \frac{-a_{11}(y_1, y_3, \tau)}{D_{12}} dy_3 \\
+ \int_0^t d\tau \int_0^{L_1} dy_1 \int_0^{H_1} G_1(x, y, t-\tau)\big|_{y_2=L_2} a_{12}(y_1, y_3, \tau)\, dy_3 \\
+ \int_0^t d\tau \int_0^{L_1} dy_1 \int_0^{L_2} G_1(x, y, t-\tau)\big|_{y_3=0} \frac{-a_{13}(y_1, y_2, \tau)}{D_{13}} dy_2,
\end{gathered} \tag{113}$$

is a solution to problem (1)-(8) at $i = 1$ under conditions $f_1(x, t) \equiv 0$, $c_{10}(x) \equiv 0$ (i.e. with homogeneous equation (1), zero initial condition (2), and non-zero boundary conditions (3)-(7));

- function $c_2(x, t)$, which is defined as

$$\begin{gathered}
c_2(x, t) = \int_0^t d\tau \int_0^{L_2} dy_2 \int_{H_1}^{L_3} G_2(x, y, t-\tau)\big|_{y_1=0} \frac{-c_{21}(y_2, y_3, \tau)}{D_{21}} dy_3 \\
+ \int_0^t d\tau \int_0^{L_2} dy_2 \int_{H_1}^{L_3} G_2(x, y, t-\tau)\big|_{y_1=L_1} a_{21}(y_2, y_3, \tau)\, dy_3 \\
+ \int_0^t d\tau \int_0^{L_1} dy_1 \int_{H_1}^{L_3} G_2(x, y, t-\tau)\big|_{y_2=0} \frac{-a_{11}(y_1, y_3, \tau)}{D_{22}} dy_3 \\
+ \int_0^t d\tau \int_0^{L_1} dy_1 \int_{H_1}^{L_3} G_2(x, y, t-\tau)\big|_{y_2=L_2} a_{22}(y_1, y_3, \tau)\, dy_3 \\
+ \int_0^t d\tau \int_0^{L_1} dy_1 \int_0^{L_2} G_2(x, y, t-\tau)\big|_{y_3=L_3} \frac{-a_{23}(y_1, y_2, \tau)}{D_{23}} dy_2,
\end{gathered} \tag{114}$$

is a solution to problem (1)-(8) at $i = 2$ under conditions $f_2(x, t) \equiv 0$, $c_{20}(x) \equiv 0$ (i.e. with homogeneous equation (1), zero initial condition (2), and non-zero boundary conditions (3)-(7)).

The mechanism for constructing formulas (113) and (114) is quite obvious: the first term in formulas (113) and (114) is a solution to problem (1)-(8) under the conditions $f_i(x, t) \equiv 0$, $c_{i0}(x) \equiv 0$, $a_{i1}(x_2, x_3, t) \equiv 0$, $c_{i2}(x_1, x_3, t) \equiv 0$, $a_{i2}(x_1, x_3, t) \equiv 0$, $a_{i3}(x_1, x_2, t) \equiv 0$ for $\forall i = 1, 2$; the second term in (113), (114) is a solution to problem (1)-(8) under the conditions $f_i(x, t) \equiv 0$, $c_{i0}(x) \equiv 0$, $c_{i1}(x_2, x_3, t) \equiv 0$, $a_{i2}(x_1, x_3, t) \equiv 0$, $c_{i2}(x_1, x_3, t) \equiv 0$, $a_{i3}(x_1, x_2, t) \equiv 0$ for $\forall i = 1, 2$; the third term in (113), (114) is a solution to problem (1)-(8) under the conditions $f_i(x, t) \equiv 0$, $c_{i0}(x) \equiv 0$, $c_{i1}(x_2, x_3, t) \equiv 0$, $a_{i1}(x_2, x_3, t) \equiv 0$, $a_{i2}(x_1, x_3, t) \equiv 0$, $a_{i3}(x_1, x_2, t) \equiv 0$ for $\forall i = 1, 2$; the fourth term in (113), (114) is a solution to problem (1)-(8) under conditions $f_i(x, t) \equiv 0$, $c_{i0}(x) \equiv 0$, $c_{i1}(x_2, x_3, t) = a_{i1}(x_2, x_3, t) \equiv 0$, $c_{i2}(x_1, x_3, t) \equiv 0$, $a_{i3}(x_1, x_2, t) \equiv 0$ for $\forall i = 1, 2$; the last term in (113), (114) is a solution to problem (1)-(8) under conditions $f_i(x, t) \equiv 0$, $c_{i0}(x) \equiv 0$, $c_{i1}(x_2, x_3, t) \equiv 0$, $a_{i1}(x_2, x_3, t) \equiv 0$, $c_{i2}(x_1, x_3, t) \equiv 0$, $a_{i2}(x_1, x_3, t) \equiv 0$ for $\forall i = 1, 2$.

Thus, the function $c_1(x, t)$, which is obtained by summing the right parts of formulas (112) (at $j = 1$) and (113), describes the desired dynamics of the concentration of metal substances in the first layer of a two-layer peat block; the function $c_2(x, t)$, which is obtained by summing the right parts of formulas (112) (at $j = 2$) and (114) describes the desired dynamics of the concentration of metal substances in the second layer of a two-layer peat block.

Remark 9. *The fundamental book Carslaw & Jaeger (1959) presents a very elegant approach, which allows in some cases to reduce initial boundary problems (with Dirichlet, Neumann, Robin or mixed types boundary conditions) for a homogeneous parabolic equation in an anisotropic medium*

$$\frac{\partial T(x,\,t)}{\partial t}=\sum_{j=1}^{3}K_{jj}\frac{\partial^2 T(x,\,t)}{\partial x_j^2}+\frac{1}{2}\sum_{i,\,j=1:\;i\neq j}^{3}(K_{ij}+K_{ji})\frac{\partial^2 T(x,\,t)}{\partial x_i\partial x_j},\;\left(K_{jj}>0\;\forall j=\overline{1,\,3}\right) \quad (115)$$

to the corresponding initial-boundary value problems for a homogeneous parabolic equation of in an isotropic medium

$$\frac{\partial T(y,\,t)}{\partial t}=K\sum_{j=1}^{3}\frac{\partial^2 T(y,\,t)}{\partial y_j^2}, \quad (116)$$

where K is an arbitrary constant; $y_j=\xi_j\sqrt{\frac{K}{K_j}},\; j=\overline{1,\,3}$ are coordinates of the transformation $y=F(x)$ allowing to move from equation (115) to the equation

$$\frac{\partial T(\xi,\,t)}{\partial t}=\sum_{j=1}^{3}K_j\frac{\partial^2 T(\xi,\,t)}{\partial \xi_j^2}; \quad (117)$$

the rectangular coordinates $\xi_j,\; j=\overline{1,\,3}$, of which are called the principal axes of equation (115), form a new coordinate system in which the quadratic form $\sum\limits_{j=1}^{3}K_{jj}x_j^2+\frac{1}{2}\sum\limits_{i,\,j=1:\;i\neq j}^{3}(K_{ij}+K_{ji})x_ix_j$ turns to the form $\sum\limits_{j=1}^{3}K_j\xi_j^2$, where $K_j,\; j=\overline{1,\,3}$ are the corresponding coefficients of this transformation.

The proposed approach is guaranteed to be applicable in the following cases:

- *if the considered area is not limited;*
- *if the considered area is bounded by planes perpendicular to the principal axes $\xi_j,\; j=\overline{1,\,3}$ of the parabolic equation (117);*
- *if $K_2=K_3$ and the considered area is bounded by planes perpendicular to the axis ξ_1 and circular cylinders whose axis coincides with the axis ξ_1.*

In most other cases, the boundaries of the considered area are distorted and, hence, anisotropy cannot be eliminated Wooster (1949). Unfortunately, the elegant approach outlined above is not applicable to the problem studied in this paper: it is also impossible to get rid of anisotropy.

The construction of the analytical solution of the problem (1)-(8) is entirely completed.

5 Conclusion

In this paper, it is studied the problem of determining the dynamics of the concentration of metal substances in a two-layer anisotropic peat block. The work examines in detail the well-known variables separation method for constructing an analytical solution for a mathematical model of the studied problem. It is shown that the main difficulty is only the solution of interrelated auxiliary problems AP1 and AP2, which are obtained from the original mathematical model under the conditions that there are no sources in both layers, and that all boundary conditions are homogeneous.

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